KEY DEFINITIONS IN SISO DESIGNS

1. The closed loop structure

\[ T \xrightarrow{} \hat{e}(s) \xrightarrow{} K(s) \xrightarrow{} d_i(s) \xrightarrow{} G(s) \xrightarrow{} d_o(s) \xrightarrow{} y(s) \xrightarrow{} n(s) \]

- \( T \): input command
- \( \hat{e}(s) \): error signal
- \( d_i(s) \): input disturbance signal
- \( n(s) \): sensor noise signal
- \( d_o(s) \): output disturbance signal
- \( y(s) \): output signal
- \( G(s) \): The plant transfer function
- \( K(s) \): Compensator transfer function

This is the transfer function we must design to meet the control problem specifications.

2. Relationships between the various variables
\[ y(s) = \frac{G(s) K(s)}{1 + G(s) K(s)} \]  
\[ u(s) \quad \text{ref} \leftrightarrow o/p \]

\[ + \frac{G(s)}{1 + G(s) K(s)} d_1(s) \quad \text{input dist} \leftrightarrow o/p \]

\[ + \frac{1}{1 + G(s) K(s)} d_0(s) \quad \text{output dist} \leftrightarrow o/p \]

\[ - \frac{G(s) K(s)}{1 + G(s) K(s)} \eta(s) \quad \text{sensor noise} \leftrightarrow o/p \]

**Notation:**
- \( G(s) K(s) \): Loop transfer function
- \( \frac{1}{1 + G(s) K(s)} \): Return difference TF
- \( \frac{G(s) K(s)}{1 + G(s) K(s)} \): Closed loop TF
- \( \frac{1}{1 + G(s) K(s)} \): Sensitivity TF

3. **Closed Loop Stability**

The closed loop system is stable if and only if the poles of the CLTF are in the open left half plane.

**Note:** Both \( G(s) \) and \( K(s) \) may be unstable, but their combination can lead to a stable closed loop system.
II. Criteria of Goodness:

Make $1 + G(s)k(s)$ "large" in certain frequency ranges which also means that the loop transfer function $G(s)k(s)$ must necessarily be large in the same frequency range or "small" in other frequency ranges. The above statements implicitly assume that the closed loop system is stable.

Note: Different signals that appear in Fig.1 tend to have most of their energy in different and often distinct frequency ranges.

Notation: Given any signal $z(t)$ with the Laplace transform $Z(s)$, we shall use the notation $s \in \Omega_z$ to mean the set of frequencies in which the signal $z(t)$ has most of its "frequency content."

II.1 Stability of a Closed-Loop System

$$g_{cc}^*(s) = \frac{G^*(s)k^*(s)}{1 + k^*(s)G^*(s)}$$

i.e. $k^*(s)$ is designed on the basis of a nominal open loop $G^*(s)$ leading to the nominal closed loop $G_{cc}^*(s)$ which is stable.
The main problem with the stability of physical closed-loop systems relates to the present modeling errors $SG(s)$ where the true plant is

$$G(s) = G^*(s) + SG(s)$$

Once the nominal compensator $K^*(s)$ is designed we must be sure that the true closed loop system

$$G_{CL}(s) = \frac{[G^*(s) + SG(s)] K^*(s)}{1 + [G^*(s) + SG(s)] K^*(s)}$$

remain stable.

Example: P.M. and G.M. are measures used for this purpose.

Note: In general, the engineer must know a certain amount of at least qualitative information about the model error $SG(s)$, and this knowledge should strongly influence how he designs $K^*(s)$.

Most of the serious modeling errors occur at high freq. and they often represent intentional modeling approximations to keep the design overly within reasonable bounds. They must be considered in the stability analysis; as we shall see, high freq. unmodeled dynamics will put a limitation...
on the bandwidth and subsequent roll off behaviour of the nominal design.

\[ e(t) = r(t) - y(t) \approx 0 \]

In the freq. domain we express this requirement as follows:

\[ \gamma(s) = r(s) \quad \text{for} \quad s \in \mathbb{R} \]

Most applications have reference inputs having most of their energy at low frequencies. Typical inputs are:

- Steps: \( r(t) = \frac{1}{s} \)
- Ramp: \( r(t) = \frac{1}{s^2} \)
- Sinusoids: \( r(t) = \frac{s}{s^2 + \omega^2} \) or \( \frac{\omega}{s^2 + \omega^2} \)

or combinations thereof.

To accomplish command following we must have

The return-diff. IF

\[ 1 + G(s) \frac{K(s)}{s} \quad "\text{large}" \quad \text{for} \quad s \in \mathbb{R} \]

\[ \Rightarrow \quad G(s) \frac{K(s)}{s} \quad "\text{large}" \quad \text{for} \quad s \in \mathbb{R} \]

If \( G(s) \) itself is large \( s \in \mathbb{R} \), then we can simply use \( K(s) = K \) \( s \gg 1 \) provided the stability is O.K.
If $G(s)$ is "small" for $s \approx s_0$, then we must introduce dynamics in $K(s)$ to satisfy the high loop gain condition.

### 2.3 Disturbance Rejection

What we mean by disturbance rejection is the desired property for the CL system to be insensitive to input disturbance signals $d_i(t)$ or output disturbance signals $d_o(t)$.

Assume $d_i(s) = 0$

and consider the error signal

$$e(s) = r(s) - y(s)$$

$$= \frac{1}{1 + G(s)K(s)} \frac{d_i(s)}{1 + G(s)K(s)}$$

$$= \frac{1}{1 + G(s)K(s)} d_o(s)$$

It is clear that disturbance rejection is accomplished if

$$1 + G(s)K(s)$$

is "large" for $s \approx s_0$, or equivalently

$$G(s)K(s)$$

is "large" for $s \approx s_0$

Thus, we need high loop gain at the disturbance frequencies, assuming that this does not violate closed loop stability.
For high loop gain, the effect of \( d_1(s) \) on \( e(s) \) is approximately
\[
e(s) \approx -\frac{1}{k(s)} \cdot d_1(s).
\]

\[
e(s) \approx -\frac{1}{G(s)k(s)} \cdot d_0(s).
\]

Thus for good input-dist. rejection, \( k(s) \) should be of the form
\[
k(s) \approx k_0 d_1(s); \quad k \gg 1 \text{ or } k_0.
\]

Note: The compensator \( k(s) \) must contain the input-dist. dynamics, and hence the loop TF must contain the input-dist. dynamics.

A similar conclusion can be reached by examining (14.3.2)

If \( G(s) \) is not large at the range of \( f_o \), then \( \Phi_{d_0} \) that the energy content of the output-dist. \( d_0(s) \) is significant, then one more one should select the compensator \( k(s) \) so that
\[
k(s) \approx k_0 d_0(s); \quad k \gg 1; \quad s \ll \Phi_{d_0}.
II. 4 Adding Integrators to the Linear System

Step, ramp, parabolic.

\[ G(s) \cdot H(s) = \frac{1}{s^N} \]

II. 5 Sensitivity of the Open-Loop System

\[ \begin{align*}
    r(s) & \quad G(s) \quad y(s) \\
    & \quad G(s) \\
\end{align*} \]

\[ y^*(s) = G^*(s) \cdot r(s) \]

Modeling errors:

\[ g(s) = G(s) + \Delta G(s) \]

Thus

\[ y(s) = G(s) \cdot r(s) \]

\[ = G^*(s) \cdot r(s) + \Delta G(s) \cdot r(s) \]

\[ = y^*(s) + \Delta G(s) \cdot r(s) \]

Define

\[ \Delta y(s) = \Delta G(s) \cdot r(s) \]

\[ \frac{\Delta y(s)}{y^*(s)} = \frac{\Delta G(s)}{G^*(s)} \]

The deviation \( \Delta y(s) \) of the output is proportional to the modeling error \( \Delta G(s) \).

\( \sigma_g \) varies with \( t \) \( \Rightarrow \) \( \sigma_y \) varies with \( y^* \).\]
Sensitivity of the closed loop system

\[ y^*(s) = \frac{G^*(s) K(s)}{1 + G^*(s) K(s)} \cdot r(s) \]

Actual: \[ G(s) = G^* + \Delta G \]
\[ y(s) = y^*(s) + \Delta y(s) \]
\[ y(s) = y^* + \Delta y = \frac{(G^* + \Delta G) K}{1 + (G^* + \Delta G) K} \cdot r(s) \]

\[ \frac{\Delta y}{y^*} = \frac{1}{1 + G(s) K(s)} \cdot \frac{\Delta G}{G^*} \]

Make \( G(s) K(s) \) large \( \\text{or} \) \( \Delta G \) small.

Thus if we have
\[ \left| \frac{1}{1 + G(s) K(s)} \right| = 100 \]
\[ \left| \frac{G(s)}{G^*} \right| = 0.1 \left| \frac{\Delta G}{G^*} \right| \]
Then \( 0.1 \Delta y \) change in \( G^*(s) \) \( \Rightarrow \) 0.18 change in \( y^* \).

Note: of course the actual closed loop must be stable.
Summary of Desirable Properties and Limitations:

Performance:

1. Good command following
2. Good disturbance rejection
3. Good sensitivity reduction to modeling errors.

Good feedback loops have large loop transfer functions, $G(s)K(s)$, and large return differences $1 + K(s)G(s)$.

Limitations of Feedback Systems:

(a) Stability/robustness with unmodeled dynamics.
(b) Sensor noise response.
(c) High freq. roll off.

$$
\int_{-\infty}^{\infty} \left| \frac{1}{s} \right| \left( 1 + G(j\omega)K(j\omega) \right) \, ds = 0
$$

This states that large return difference transfer functions at some frequency ranges must be balanced by small return difference transfer functions at other frequencies.

Typically, we require at least 2 more poles than zeros.
The challenge is then:

Given an open-loop transfer function $G(s)$, find a compensator $H(s)$ which leads to a stable closed-loop system and its loop transfer function has the properties shown below.

---

Note: low freq. freq. objectives, high freq. limitations.
Short Course

Robust Multivariable Control System

Analysis and Design

January 10-11, 1991

NASA Johnson Space Center
Houston, Texas

Speakers

Alexander G. Parlos & Amir Atiya
Texas A&M University
College Station, Texas 77843
Course Outline

JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
- HISTORICAL OVERVIEW
- MATHEMATICAL BACKGROUND
- UNCERTAINTY MODELING
- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
- LQR, KF, LQG
- LQ-Servo, LQG/LTR
JANUARY 11, 1991:

- $H_\infty$ - OPTIMAL SYNTHESIS PROBLEM
- SOLUTION TO $H_\infty$ - OPTIMAL FULL STATE FEEDBACK CONTROL PROBLEM
- MULTIVARIABLE DESIGN USING $H_\infty$ APPROACH
- ADDRESSING ROBUST PERFORMANCE AND THE $\mu$ SYNTHESIS
- SUMMARY AND REMARKS
JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
- HISTORICAL OVERVIEW
- MATHEMATICAL BACKGROUND
- UNCERTAINTY MODELING
- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
- LQR, KF, LQG
- LQ-Servo, LQG/LTR
## Control Engineering

### Systems to be Controlled

- **Mechanical**
  - Rigid Body
  - Flexible

- **Electrical/Electronic**

- **Electro-Mechanical**

- **Process**
  - Chemical/Petrochemical
  - Manufacturing

### Control Design Tools

- Classical Design
  - Linear
    - LQR, LQG
    - LQR, LTR - H_\infty
  - MPC
    - Adaptive
    - Variable Structure
    - Feedback Linearization

- Model-Based
  - PID
  - On-line Optimization
Introduction & Objectives

• **THE COURSE IS:**
  • A BRIEF EXPOSURE OF THE ANALYSIS AND DESIGN ISSUES IN MULTIVARIABLE (PREFERABLY ROBUST) CONTROL SYSTEMS.
  • SPECIAL EMPHASIS ON A CERTAIN TOOL IN ACHIEVING MULTIVARIABLE DESIGN: $H_\infty$ - OPTIMIZATION.

• **THE COURSE IS NOT:**
  • ROBUST CONTROL THEORY COURSE,
  • A SURVEY OF DESIGN TECHNIQUES.
WHAT IS A ROBUST CONTROL SYSTEM:

A CONTROL SYSTEM IS SAID TO BE ROBUST IF IT EXHIBITS SATISFACTORY PERFORMANCE AND STABILITY (MARGINS) IN THE PRESENCE OF PLANT (MODEL) UNCERTAINTY.

WHY CARE ABOUT ROBUST CONTROL SYSTEMS:

FOR THE SAME REASON THAT CLOSED-LOOP FEEDBACK CONTROL WAS THOUGHT IMPORTANT IN CLASSICAL CONTROL:

MODELING UNCERTAINTY

REASON FOR DOING THINGS DIFFERENTLY THAN IN CLASSICAL CONTROL:

MULTI-INPUT MULTI-OUTPUT SYSTEMS
INTRODUCTION & OBJECTIVES (Cont'd)

CONTROLLER CLASSES

NONLINEAR

ADAPTIVE

GAIN SCHEDULED

FIXED GAIN

ROBUST
Course Outline

JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
- HISTORICAL OVERVIEW
- MATHEMATICAL BACKGROUND
- UNCERTAINTY MODELING
- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
- LQR, KF, LQG
- LQ-Servo, LQG/LTR
### Historical Overview

**A Timetable for Linear Control System Design**

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Historical Overview (Cont'd)

• REFERENCES:
  • A COURSE IN $H_{\infty}$ CONTROL THEORY, B. FRANCIS, 1987.
  • ROBUST PROCESS CONTROL, M. MORARI, ET. AL., 1989.
  • FEEDBACK CONTROL THEORY, J. DOYLE, ET. AL., 1991 ??

• TOOLS:
  • MATRIXx
  • MATLAB, ROBUST CONTROL TOOLBOX AND MORE.
JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
- HISTORICAL OVERVIEW
- MATHEMATICAL BACKGROUND
- UNCERTAINTY MODELING
- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
- LQR, KF, LQG
- LQ-Servo, LQG/LTR
Motivating the Mathematics

IF $|u(s)|$ IS LARGE AND WE WANT $|y(s)|$ SMALL WE NEED TO MAKE $|G(s)|$ SMALL

IF "u(s)" IS LARGE AND WE WANT "y(s)" SMALL WE STILL NEED TO MAKE "G(s)" SMALL

WHAT DOES LARGE (SMALL) $u(s)$ ($y(s)$) MEAN?

WHAT DOES LARGE OR SMALL $G(s)$ MEAN?
Mathematical Background

- VECTOR NORMS
- MATRIX NORMS
- SINGULAR VALUES AND SINGULAR VALUE DECOMPOSITION (PRINCIPAL GAINS)
  (Example)
- STATE SPACE TO TRANSFER FUNCTION TRANSFORMATIONS
- LINEAR FRACTIONAL TRANSFORMATIONS
- SCALAR SIGNAL NORMS
- TRANSFER FUNCTION NORMS
- COMPUTATION OF $\infty$ - NORM
  (Example)
Useful References

(I) SISO Frequency design methods.

(II) System & MIMO systems.

(III) Frequency domain design methods for MIMO systems:
BASIC DEFINITIONS & ISSUES IN MULTIVARIABLE FEEDBACK SYSTEMS

We are interested in examining properties of multi-input multi-output (MIMO) linear time invariant (LTI) systems. The basic approach follows closely the one outlined in SISO systems as far as command-following, disturbance rejection, sensitivity reduction etc. is concerned. However, problems arise when systematic approaches are needed to satisfy a set of given design specs. In addition, rules of thumb such as "large loop gains" can not be used unless additional definitions are made for concepts such as "magnitude of a matrix". That is even though we can measure the largeness of a scalar, and form a vector, it is not straightforward to do the same for an arbitrary matrix i.e. a linear map.

Therefore, we will introduce the concept of a singular value (SV) of a general complex matrix in order to quantify largeness.

1. The Standard MIMO Feedback Loop

As in the case of a SISO feedback loop, the MIMO loop is given as shown below:
Definitions:

\[ r(s) \in \mathbb{R}^m : \text{input command/reference vector} \]
\[ e(s) \in \mathbb{R}^m : \text{error vector} \]
\[ d_i(s) \in \mathbb{R}^m : \text{input disturbance vector} \]
\[ d_o(s) \in \mathbb{R}^m : \text{output disturbance vector} \]
\[ y(s) \in \mathbb{R}^m : \text{output vector} \]
\[ n(s) \in \mathbb{R}^m : \text{sensor noise vector} \]

\[ G(s) : \text{m} \times \text{m} \text{ open-loop transfer matrix} \]

\[ K(s) : \text{m} \times \text{m} \text{ compensator transfer matrix} \]

Designed to meet a given set of design specs. Usually \( K(s) \) is designed in the analog world and implemented digitally by microprocessors.

It is assumed that all vectors are of \( m \) dimension. This does not have to be like this, however, if

\[
\# \text{ of inputs} > \# \text{ of outputs to be controlled in } k
\]

then there is a redundancy in the system and controls can be reduced. If \( k < m \) then we can only control \( m - k \) out of the \( m \) outputs independently.
To derive the closed-loop transfer function matrix proceed as in the SISO case:

Let: \[ e(s) = r(s) - y(s) - y'(s) \]  \hspace{0.5cm} \text{(1)}

where \[ y(s) = d_0(s) + G(s) d_1(s) + G(s) K(s) e(s) \]  \hspace{0.5cm} \text{(2)}

by eliminating \( e(s) \) from (2) by using (1), we have:

\[ y(s) = \left[ I + G(s) K(s) \right]^{-1} G(s) K(s) e(s) + \]

\[ \left[ I + G(s) K(s) \right]^{-1} d_0(s) + \]

\[ \left[ I + G(s) K(s) \right]^{-1} G(s) d_1(s) - \]

\[ \left[ I + G(s) K(s) \right]^{-1} G(s) K(s) y(s) \]  \hspace{0.5cm} \text{(3)}

Also obtain the expression for the error:

\[ e(s) = \left[ I + G(s) K(s) \right]^{-1} e(s) - \left[ I + G(s) K(s) \right]^{-1} d_0(s) - \]

\[ \left[ I + G(s) K(s) \right]^{-1} G(s) d_1(s) - \left[ I + G(s) K(s) \right]^{-1} y(s) \]  \hspace{0.5cm} \text{(4)}

Definitions:
- \( G(s) K(s) \): Loop Transfer matrix
- \( I + G(s) K(s) \): Return Difference Transfer matrix
- \( [I + G(s) K(s)]^{-1} \): Sensitivity T.F.M.
Sensitivity Relations

As in the $S_{150}$ case, assume that the nominal plant is denoted by $\gamma^*(s)$ and also set,

$\delta_0(s) = d$; $r(s) = y(s) = 0$.

The nominal plant output is:

$$ g^*(s) = \left[ I + G^*(s) K(s) \right]^{-1} G^*(s) K(s) r(s) \quad (5) $$

Now, assume,

$$ G(s) = G^*(s) + \delta G(s) $$

$\delta G(s)$ will result in a change $\delta y(s)$ at the plant output:

$$ y(s) = y^*(s) + \delta y(s) = $$

$$ = \left[ I + [G^*(s) + \delta G(s)] K(s) \right]^{-1} [G^*(s) + \delta G(s)] K(s) r(s) \quad (6) $$

Using (5) and (6) we obtain:

$$ \delta y(s) = \left[ I + G(s) K(s) \right]^{-1} \delta G(s) \left[ G^*(s) \right]^{-1} y^*(s) \quad (7) $$
2. Properties of NIRO Feedback Systems

In addition to closed-loop stability we usually ask the following questions relevant to the performance of a feedback loop:

1. Command Following
2. Disturbance Rejection
3. Sensitivity Reduction
4. Sensor Noise Response

We will translate these design requirements to issues relating the magnitude of a transfer function matrix.

(1) Command Following:

For good command following we want

\[ \gamma(s) \approx 0 \text{ for all } \sigma \geq 0 \]

That is, for all frequencies at which reference signals have most of their energy. From the \( \gamma(s) \to z(s) \) relation it can be seen that good command following can be achieved if

\[ [I + G(s)K(s)]^{-1} G(s)K(s) \approx I \text{ for } s \in \sigma \]

(8) Good command following also requires that \( e(s) = 0 \) for \( s \geq 0 \). However, this can be accomplished if

\[ [I + G(s)K(s)]^{-1} = 0 \text{ for } s \geq 0 \]  

(8)
Even though we have not defined the magnitude of a matrix we can say that in order for (2) to be valid,

\[ G(s)K(s) \text{ must be "large" such that } \]

\[ [I + G(s)K(s)] = G(s)K(s) \]

If this is true (3) is also satisfied.

So for good command following we would like to have the loop transfer function matrix large in the frequency range of interest, as far as the command signals are concerned.

(2) Disturbance Rejection

From equation (3) it is apparent that \([I + G(s)K(s)]^{-1}\) has to be small in order to have good disturbance rejection. This requires

\[ G(s)K(s) \text{ to be "large" at the appropriate frequency range relevant to disturbances} \]

(3) Sensitivity Reduction

From equation (7) it is clear that in order to minimize the effects of \(E(s)\) on \(E(s)\) we should have large \(G(s)K(s)\) at the frequency range
that $S(s)$ is significant.

4. Sensor Noise

Again from equation (3) we can see that in order to minimize the effect that sensor noise has on the plant output we should have:

"small" $[I + G(s)K(s)]G(s)K(s)$ in the range of frequencies that $S(s)$ is significant.

We have seen that in order to design a MIMO feedback loop with good performance characteristics we need to:

1. Stabilize the closed-loop system
2. Quantify the magnitude of the loop transfer matrix $G(s)K(s)$ and the return difference $I + G(s)K(s)$

3. Example for matrix magnitudes

Consider a $2 \times 2$ system, 2 inputs & 2 outputs and assume that

$$G(s) = \begin{bmatrix}
\frac{10^6}{5} & 0 \\
0 & \frac{10^{-6}}{5}
\end{bmatrix}$$
Even though decoupled, the system is a \( m \times n \) type.

We try to find the magnification caused to an input signal that goes through the system. Assume \( H(s) = \frac{3}{s^2 + \omega^2} \), i.e., \( u(t) \) is a sinusoid.

And suppose that, \( u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), then the output vector \( y(s) = G(s)u(s) \) is:

\[
y(s) = \frac{\begin{bmatrix} 10^6 \\ 0 \end{bmatrix}}{s^2 + \omega^2}
\]

Thus, the output is amplified by a factor of \( 10^6 \). Thus, we would say that the \( G(s) \) is "large."

Now, assume \( u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), the,

\[
y(s) = \frac{\begin{bmatrix} 0 \\ 10^{-6} \end{bmatrix}}{s^2 + \omega^2}
\]

So, the output is attenuated by a factor of \( 10^{-6} \); and we would conclude that \( G(s) \) is small.

**Conclusion**: We cannot judge the size of a matrix without looking at all possible directions of the input. Actually, the above matrix is small because it attenuates the input in a particular direction. In order for \( G(s) \) to be large, it must
4. **Norm of Complex Vectors, and Matrices & Matrix Singular Values**

Assume, \( \alpha = a + jb \). Then, \( x^* = a - jb \), and

\[
|x| = \sqrt{x^*x} = \sqrt{a^2 + b^2}
\]

Now, consider \( x = [x_1, \ldots, x_n] \) where \( x_i \) is a complex scalar. Then, the generalization of the magnitude of the vector \( x \) is its Euclidean norm, denoted by \( \|x\|_2 \), and defined by:

\[
\|x\|_2 = \left[ \sum_{i=1}^{n} |x_i|^2 \right]^{1/2}
\]

Given a complex column vector \( x \), we define \( x^H \) to be the row vector whose elements are the complex conjugates of the elements of \( x \). For example,

\[
x = \begin{bmatrix} 2 + j3 \\ 4 - j2 \end{bmatrix} \quad \text{then} \quad x^H = \begin{bmatrix} 2 - j3 \\ 4 + j2 \end{bmatrix}
\]

Then,

\[
x^Hx = \sum_{i=1}^{n} |x_i|^2 \quad \text{is a scalar product of} \ x \ \text{with itself} \ i^* \ \text{and},
\]

\[
\|x\|_2 = \sqrt{x^Hx}
\]
• VECTOR NORMS

\[ \| x \|_1 = \sum_{i=1}^{n} |x_i| \]

\[ \| x \|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \quad \text{(Euclidean Norm)} \]

\[ \| x \|_\infty = \max_{i} |x_i| \]
Now, consider an $n \times n$ complex matrix $A$. Then,

$$A^H = \text{complex conjugate transpose of } A.$$ 

**Definition 1.** A complex matrix $A$ is hermitian if $A = A^H$.

**Definition 2.** A complex matrix $U$ is unitary if $U^H = U^{-1}$.

Some properties:

**Property 1:** All the eigenvalues of a hermitian matrix are real.

**Property 2:** All of the eigenvalues of a unitary matrix have unit magnitude.

**Property 3:** If $A$ is a hermitian matrix then there exists a unitary matrix $U$ such that:

$$A = U \Lambda U^H$$

where $\Lambda$ is the diagonal matrix of the eigenvalues of $A$. The columns of $U$ are the eigenvectors of $A$, and they are orthogonal.
Property 4: (Rayleigh's quotient) If $A$ is hermitian then,

$$\min_{x \neq 0} \frac{x^* A x}{x^* x} = \lambda_{\min}(A)$$

where, $\lambda_{\min}(A)$ is the minimum eigenvalue of $A$. The minimum is attained when $x$ is the eigenvector of $A$ corresponding to $\lambda_{\min}(A)$.

Also,

$$\max_{x = 0} \frac{x^* A x}{x^* x} = \lambda_{\max}(A)$$

where, $\lambda_{\max}(A)$ is the maximum eigenvalue of $A$. The maximum is attained when $x$ is the eigenvector of $A$ corresponding to $\lambda_{\max}(A)$.

Now we define the spectral norm, $\|A\|_2$ of any complex matrix $A$ (not necessarily hermitian) by:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

It turns out that:

$$\|A\|_2 = \sqrt{\max_i \lambda_i(A^* A)} ; \quad i = 1, \ldots, n.$$ 

Comment: Note that $A^* A$ is Hermitian and positive semi definite; hence, the eigenvalues of $A^* A$, $\lambda_i(A^* A)$, are real and non-negative. If $A$ is nonsingular, $A^* A$ is positive definite, and $\lambda_i(A^* A)$ are positive.
for all \( i = 1, 2, \ldots, n \).

Now let's introduce the concept of a singular value of an \( n \times n \) complex matrix \( A \), denoted by \( \sigma_i(A) \), \( i = 1, \ldots, n \). The singular values are defined by:

\[
\sigma_i(A) = \sqrt{\lambda_i(A^*A)} \geq 0; \quad i = 1, \ldots, n
\]

and they are all non-negative.

It turns out that:

\[
\sigma_{\text{max}}(A) = \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} = ||A||_2
\]

and

\[
\sigma_{\text{min}}(A) = \min_{x \neq 0} \frac{||Ax||_2}{||x||_2} = \frac{1}{||A^{-1}||_2}
\]

provided that \( A^{-1} \) exists.

Therefore, the maximum singular value of \( A \), \( \sigma_{\text{max}}(A) \), is simply the spectral norm of \( A \). The spectral norm of \( A^{-1} \) is the inverse of the \( \sigma_{\text{min}}(A) \), the minimum singular value of \( A \).

So, in view of the above definitions:

\[
\sigma_{\text{max}}(A^{-1}) = \frac{1}{||A^{-1}||_2} = \frac{1}{\sigma_{\text{min}}(A)}
\]
Mathematical Background (Cont'd)

- **MATRIX NORMS**

\[
\| A \|_p = \sup_{x \neq 0} \frac{\| Ax \|_p}{\| x \|_p}
\]

\[
\| A \|_1 = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(Maximum "column sum")}
\]

\[
\| A \|_{\infty} = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(Maximum "row sum")}
\]

\[
\| A \|_2 = \sigma_{\max}(A) \quad \text{(Maximum singular value)}
\]
Mathematical Background (Cont'd)

- SINGULAR VALUES AND SINGULAR VALUE DECOMPOSITION (PRINCIPAL GAINS)

"THE SQUARE OF SINGULAR VALUES OF A ARE THE EIGENVALUES OF $A^T A$.

$$A = U \Sigma V^T \quad \text{WHERE,} \quad \Sigma = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_p)$$

$$\| A \|_2 = \bar{\sigma} (A) = \sigma_1 \quad \bar{\sigma} (A) = \frac{1}{\sigma (A^{-1})}$$
Mathematical Background (Cont'd)

- **TRANSFER FUNCTION NORMS**

\[
\left\| G \right\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr} \left[ G(j\omega)^* G(j\omega) \right] d\omega = \int_{-\infty}^{+\infty} \left\| g(t) \right\|_2^2 dt
\]

\[
\max \quad \left\| G \right\|_\infty = \text{ess sup} \sigma(G(j\omega))_{\omega \in \mathbb{R}}
\]

\[
\left\| G \right\|_\infty = \max \sup_{u \neq 0} \frac{\left\| Gu \right\|_2}{\left\| u \right\|_2} \quad \Rightarrow \quad \left\| G \right\|_\infty = \max \sup_{\omega \in \mathbb{R}} \sigma(G(j\omega))
\]
\[ \sigma_{\min}(A^{-1}) = \frac{1}{\|A\|_2} = \frac{1}{\sigma_{\text{max}}(A)} \]

Now let's introduce the singular-value decomposition (SVD). Given a \(m \times n\) complex matrix \(A\), then there exist unitary matrices \(U\) and \(V\) such that

\[ A = U \Sigma V^H = \sum_{i=1}^{\min(m,n)} \sigma_i(A) u_i v_i^H \]

where \(\Sigma\) is a diagonal matrix of the singular values, \(\sigma_i(A)\), the \(u_i\) are the column vectors of \(U\), i.e.

\[ U = [u_1, \ldots, u_n] \]

and the \(v_i\) are the column vectors of \(V\), i.e.

\[ V = [v_1, \ldots, v_n] \]

The \(v_i\) are called the right singular vectors of \(A^* A\), because

\[ A^* A v_i = \sigma_i^2(A) v_i \]

The \(u_i\) are called the left singular vectors of \(A^* A\), because

\[ u_i^* A A^* = u_i^* \sigma_i^2(A) \]

Now we can talk about the "largeness" of a matrix.
Mathematical Background (Cont'd)

- **EXAMPLE**

\[
A = \begin{bmatrix} 0.96 & 1.72 \\ 2.28 & 0.96 \end{bmatrix} = U\Sigma V^T \\
= \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}^T
\]

**Unit Circle at Matrix Input**

**Image of Unit Circle:** \( y = Ax \)
\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]

If \( v_1, v_2 \) are on a unit circle; i.e. \( |v_1| = 1 \) & \( |v_2| = 1 \) then what is \( y_i, \text{max} \) & \( y_j, \text{min} \)
5. The minimum and maximum Singular Values of a Transfer Matrix

Consider a system with input vector \( x(s) \), output vector \( y(s) \), and \( F(s) = G(s) \) \((m \times m)\) such as:

\[
y(s) = G(s) u(s)
\]

Take the complex conjugate transpose:

\[
y^H(s) = u^H(s) G^H(s)
\]

then,

\[
y^H(s) y(s) = u^H(s) G(s) G^H(s) u(s)
\]

which implies that

\[
\| y(s) \|_2^2 = \| G(s) u(s) \|_2^2
\]

\[
\| y(s) \|_2 = \| G(s) u(s) \|_2
\]

Divide both sides by \( \| y(s) \|_2 \):

\[
\frac{\| y(s) \|_2}{\| u(s) \|_2} = \frac{\| G(s) u(s) \|_2}{\| u(s) \|_2}
\]

Analogous to \((5.150)\):

\[
\left| \frac{y(s)}{u(s)} \right| = \left| g(s) \right|
\]
In the SISO system, the magnitude of the RHS depends on the direction of \( y(s) \). The largest magnitude that the RHS can obtain can be calculated from the max. s.v. of \( G(s) \), \( \sigma_{\text{max}}(G(s)) \)

as:

\[
\max_{u(s) \neq 0} \frac{\| G(s)y(s) \|_2}{\| u(s) \|_2} = \sigma_{\text{max}}(G(s)) = \sqrt{\lambda_{\text{max}}(G^T(s)G(s))}
\]

The min. value can be calculated by the \( \sigma_{\text{min}}(G(s)) \)

\[
\min_{u(s) \neq 0} \frac{\| G(s)y(s) \|_2}{\| u(s) \|_2} = \sigma_{\text{min}}(G(s))
\]

Thus it is evident that if we want to call \( G(s) \) large for all possible inputs \( y(s) \), then we must have \( \sigma_{\text{min}}(G(s)) \) large. This minimum is attained when the input \( y(s) \) is the right singular vector associated with \( \sigma_{\text{min}}(G(s)) \).

Conversely, if we want to call \( G(s) \) small for all inputs \( y(s) \), then we must have \( \sigma_{\text{max}}(G(s)) \) be small. Note that this max. is attained when \( y(s) \) is the right singular vector associated with \( \sigma_{\text{max}}(G(s)) \).
Mathematical Background (Cont'd)

- COMPUTATION OF $\infty$-NORM

GIVEN, $G(s) = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ FORM THE FOLLOWING MATRIX

$$M_\gamma = \begin{bmatrix} A - BR^{-1}D^T & -\gamma BR^{-1}B^T \\ \gamma C^T S^{-1}C & -(A - BR^{-1}D^T)^T \end{bmatrix}$$

$$R = (D^TD - \gamma^2 I)$$

$$S = (DD^T - \gamma^2 I)$$

IF $\|G\|_\infty = \gamma_0$ THEN \(\begin{\cases} \gamma < \gamma_0 \quad M_\gamma & \text{HAS IMAG. EIGENVALUES} \\
\gamma \geq \gamma_0 \quad M_\gamma & \text{HAS REAL OR COMPLEX EIGENVALUES} \end{\cases}\)
Mathematical Background (Cont'd)

- EXAMPLE

\[ |G|_\infty \]

\[ \sigma(G) \]

0

\[ \omega \]
• STATE SPACE TO TRANSFER FUNCTION TRANSFORMATIONS

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[\iff\]

\[
G(s) = \begin{bmatrix}
\frac{B}{C} \\
\frac{A}{D}
\end{bmatrix}
\]

\[
(I - G)^{-1} = \begin{bmatrix}
A + B(I - D)^{-1}C & B(I - D)^{-1} \\
(I - D)^{-1}C & (I - D)^{-1}
\end{bmatrix}
\]
Mathematical Background (Cont'd)

\[ G(I - G)^{-1} = \left[ \begin{array}{c|c} A + B(I - D)^{-1}C & B(I - D)^{-1} \\ \hline (I - D)^{-1}C & (I - D)^{-1}D \end{array} \right] \]

\[ G_1(I - G_2G_1)^{-1} = \]

\[ \left[ \begin{array}{ccc} A_1 + B_1FD_2C_1 & B_1FC_2 & B_1F \\ B_2EC_1 & A_2 + B_2ED_1C_2 & B_2ED_1 \\ EC_1 & ED_1C_2 & ED_1 \end{array} \right] \]

where \( E = (I - D_1D_2)^{-1} \) and \( F = (I - D_2D_1)^{-1} \).
Mathematical Background (Cont'd)

- LINEAR FRACTIONAL TRANSFORMATIONS

\[ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \]

\[ z(s) = F_1(P(s), K(s)) \cdot w(s) \]
\[ z(s) = F_u(P(s), \Delta(s))w(s) \]

\[ F_u(P, \Delta) = \begin{bmatrix}
A_1 + B_1FD_3C_1 & B_1FC_3 & B_2 + B_1FD_3D_{12} \\
B_3EC_1 & A_3 + B_3ED_11C_3 & B_3ED_{12} \\
C_2 + D_{21}FD_3C_1 & D_{21}FC_3 & D_{22} + D_{21}FD_3D_{12}
\end{bmatrix} \]

where \( E = (I - D_{11}D_3)^{-1} \), \( F = (I - D_3D_{11})^{-1} \), and \( \Delta = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} \)
Mathematical Background (Cont'd)

- SCALAR SIGNAL NORMS

\[ \| g \|_1 = \int_{-\infty}^{+\infty} |g(t)| \, dt \]

\[ \| g \|_2 = \left( \int_{-\infty}^{+\infty} |g(t)|^2 \, dt \right)^{1/2} \]

\[ \| g \| = \sup_{t \in \mathbb{R}} |g(t)| \]
Additional HW. Problem. (30 points)

Calculate the $\infty$-norm of

\[ G(s) = \begin{bmatrix}
\frac{10^6}{s^2+4} & \frac{1}{s^2+9} \\
\frac{1}{s^2+9+25} & \frac{10^{-6}}{s^2+4}
\end{bmatrix} \]
MIMO Loop Shaping Concepts

By using the minimum and maximum singular values of the loop transfer matrix and of the return difference matrix, we can discuss in a quantitative manner the frequency domain loop shaping concepts for MIMO systems, simply by plotting these singular values vs. frequency on a Bode-like diagram.

As mentioned before, for good command following and disturbance rejection, will occur when the inverse return difference transfer matrix \([I + G(s)K(s)]^{-1}\) is small. This requires that

\[ \sigma_{\text{max}} \left[ [I + G(s)K(s)]^{-1} \right] \ll 1 \]

But also,

\[ \sigma_{\text{max}} \left[ [I + G(s)K(s)]^{-1} \right] = \frac{1}{\sigma_{\text{min}} \left[ [I + G(s)K(s)] \right]} \]

or, equivalently,

\[ \sigma_{\text{min}} \left[ [I + G(s)K(s)] \right] \gg 1 \]

Thus, good command following and good disturbance rejection require that the minimum singular value of the return difference transfer matrix, \(I + G(s)K(s)\), be large at
the frequency range in which the inputs and disturbances have their energy.

Using the following inequality:

$$\sigma_{\text{min}}(A) - 1 \leq \sigma_{\text{min}}(I + A) \leq \sigma_{\text{min}}(A) + 1$$

can be used to derive,

$$\sigma_{\text{min}}\left[\begin{bmatrix} G(s) & K(s) \end{bmatrix} - 1 \right] \leq \sigma_{\text{min}}\left[\begin{bmatrix} I + G(s)K(s) \end{bmatrix}\right] \leq \sigma_{\text{min}}\left[G(s)K(s)\right] + 1$$

So, we conclude that,

$$\sigma_{\text{min}}\left[G(s)K(s)\right] >> 1$$

Thus, we can conclude that for good command-following and disturbance rejection in MIMO design, it is necessary that the minimum singular value of the loop transfer matrix $G(s)K(s)$ must be large in the frequency range in which the inputs and disturbances have their energy.
Now, let's see the contribution of the sensor noise $n(s)$ to the output $y(s)$; we know that

$$y(s) = \left[ I + G(s) K(s) \right]^{-1} G(s) K(s) \eta(s)$$

Thus to minimize the effects of sensor noise, the transfer matrix $\left[ I + G(s) K(s) \right]^{-1} G(s) K(s)$ must be "small" in the range of frequencies (usually high) where the sensor noise has most of its energy. This implies that

$$\sigma_{\text{max}} \left[ \left( I + G(s) K(s) \right)^{-1} G(s) K(s) \right] =$$

$$= \| \left( I + G(s) K(s) \right)^{-1} G(s) K(s) \|_2 \ll 1$$

But,

$$\| \left( I + G(s) K(s) \right)^{-1} G(s) K(s) \|_2 \leq \| \left( I + G(s) K(s) \right)^{-1} \|_2 \cdot \| G(s) K(s) \|_2$$

$$= \sigma_{\text{max}} \left[ \left( I + G(s) K(s) \right)^{-1} \right] \sigma_{\text{max}} \left[ G(s) K(s) \right]$$

$$= \frac{\sigma_{\text{max}} \left[ G(s) K(s) \right]}{\sigma_{\text{min}} \left[ I + 2G(s) K(s) \right]}$$

So, if we make the numerator small, then we can achieve the desired goal. That is,

$$\sigma_{\text{max}} \left( G(s) K(s) \right) \ll 1$$
We can now give a MIMO statement of what we mean by loop shaping.

Let's assume that the energy of the command reference signals and of the disturbances is in the low frequency range while the sensor noise and unmodeled dynamics have energy in the high frequency range. We must then design the compensator $K(s)$ so that if we plot

$$\sigma_{\text{max}} \left[ G(s)K(s) \right] \text{ and } \sigma_{\text{min}} \left[ G(s)K(s) \right]$$

vs. frequency $\omega$,

the min. singular value of the loop transfer matrix $\sigma_{\text{min}} \left[ G(s)K(s) \right]$ is larger than the low frequency barrier that dictates command following and disturbance rejection performance while the max. singular value of the loop transfer matrix $\sigma_{\text{max}} \left[ G(s)K(s) \right]$ must be smaller than the high frequency barrier that defines in sensitivity to high frequency noise (and which also prevents the excitation of unmodeled high frequency dynamics).

In this manner the min. & maximum singular values of the loop transfer matrix are the appropriate loop shaping quantities in MIMO design; they generalize the notion of the magnitude of the loop transfer function $|T(s)|$ frequency in SISO designs.
Bode-like plot of the max. and min. singular values of the loop transfer matrix.
We remark that the shaping of the singular values in MIMO systems is a highly nontrivial affair and ad-hoc methods for determining the MIMO system's compensator H(s) to achieve a desirable min. of maximum singular value plot vs. frequency, are extremely cumbersome. What is needed is a systematic EAD procedure (SVD) that accomplishes this frequency domain loop shaping process in an easy manner, while maintaining closed-loop stability. As it will be seen later it is the cause the EAD approach accomplishes this.
SISO Zeros

"Zeros represent absorbing frequencies."

\[
\frac{s+1}{s(s+2)} \rightarrow y(t)
\]

y(t) could be made zero by excitation by the appropriate frequencies.

MIMO Systems

\[
\begin{align*}
u_1 & \rightarrow \frac{1}{s} \rightarrow Y_1 \\
u_2 & \rightarrow \frac{s+5}{s^2} \rightarrow Y_2
\end{align*}
\]

- What MIMO zeros are not?

The zeros of the individual SISO T.F have little to do with the MIMO zeros.
Towards the MIMO Poles, Zeros, Modes, Stabilizability and Detectability

For SISO systems:

\[
\begin{align*}
\dot{x}(t) &= A \cdot x(t) + B \cdot u(t) \\
y(t) &= C \cdot x(t) + D \cdot u(t)
\end{align*}
\]

and \(x(s) = L(x(t))\) then,

\[
y(s) = \left[ \frac{C}{sI - A} \right] b + d \cdot u(s)
\]

For MIMO systems:

\[
\begin{align*}
\dot{x}(t) &= A \cdot x(t) + B \cdot u(t) \\
y(t) &= C \cdot x(t) + D \cdot u(t)
\end{align*}
\]

\[
y(s) = \left[ \frac{C}{sI - A} \right] B + D \cdot u(s)
\]

Concept: Define "poles" and "zeros" in the time-domain. Poles and zeros of \((A, B, C, D)\).

The purpose of this section is to present the key ideas behind the definition of the transmission zeros for a multivariable linear time-invariant system starting from its state-space description.
Before we start the discussion of multivariable zeros it is helpful to define the generalized eigenvalue problem. As it will be seen later, the solution to this problem provides an algorithm for calculating the transmission zeros.

The Ordinary Eigenvalue Problem

Let $\mathbf{A}$ be an $m \times m$ matrix. Then the eigenvalues $\lambda_i$, right eigenvectors $\mathbf{x}_i$, and the left eigenvectors $\mathbf{y}_i$ of $\mathbf{A}$ are defined by the ordinary eigenvalue problem,

$$ (\lambda_i \mathbf{I} - \mathbf{A}) \mathbf{x}_i = \mathbf{0} ; \quad i = 1, \ldots, n $$

$$ \mathbf{y}_i^H (\lambda_i \mathbf{I} - \mathbf{A}) = \mathbf{0} ; \quad i = 1, \ldots, n $$

Clearly the eigenvalues $\lambda_i$ are the $n$ roots of the polynomial,

$$ \det(\lambda \mathbf{I} - \mathbf{A}) = 0 $$
**Uniformed System:**

\[ \dot{x}(t) = A \cdot x(t), \quad x(0) = \xi \]

Then,

\[ x(t) = e^{At} \xi \]

\[ e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k \]

\[ \xi^0(x(t)) = s \cdot x(s) - \xi \]

\[ s \cdot x(s) - \xi = A \cdot x(s) \implies x(0) = (sI - A)^{-1} \xi \]

\[ \therefore \quad \xi^0(e^{At}) = (sI - A)^{-1} = \Phi(s) \]

*the state transition matrix.*

**FACT:**

\[ e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} \cdot \bar{X}_i \cdot \bar{Y}_i \]

\[ A = \sum_{i=1}^{n} \lambda_i \cdot \bar{X}_i \cdot \bar{Y}_i^H \]

So,

\[ \xi^0(e^{At}) = (sI - A)^{-1} = \sum_{i=1}^{n} \xi^0(e^{\lambda_i t}) \cdot \bar{X}_i \cdot \bar{Y}_i^H = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} \cdot \bar{X}_i \cdot \bar{Y}_i^H \]
Observability, Detectability

Consider the following system:
\[
\dot{x}(t) = Ax(t),
\]
\[
y(t) = Cx(t).
\]
\[
C = \begin{bmatrix}
e_1 \rightarrow \\
\vdots \\
e_n \rightarrow
\end{bmatrix}
\]
\[
y_k(t) = C_k x(t).
\]
\[
x(t) = e^{At} \frac{x(0)}{x(0)},
\]
\[
= \sum_{i=1}^{n} e^{\lambda_i t} x_i \left( \frac{y_i^H}{y_i^H} \right)
\]
\[
\Rightarrow y_k(t) = \sum_{i=1}^{n} e^{\lambda_i t} \left( C_k x_i \right) \left( \frac{y_i^H}{y_i^H} \right)
\]

\((C_k \cdot x_i)\) is the degree that the \(i\)th mode shows up in the \(k\)th output (sensor).

If \((C_k \cdot x_i) = 0\) then the \(i\)th mode is unobservable in the \(k\)th output.

If \((C_k \cdot x_i) = 0, \forall k = 1, \ldots, m\)
or,
\[
y_k^H = 0 \text{ then, } \text{ith mode is unobservable.}
\]
and therefore the pair \([A, C]\) is unobservable.
or, rank \(\begin{bmatrix} C \mid AC \mid A^2C \mid \ldots \mid A^{m-1}C \end{bmatrix} < m\)

If all of the unobservable modes are stable
then \([A, C]\) is detectable.

Now, rewrite \(x(s)\) and \(y(s)\).

\[
x(s) = \sum_{i=1}^{n} \left( \frac{1}{s - \lambda_i} \right) x_i \left( \frac{y_i^H B}{s - \lambda_i} \right) + \sum_{i=1}^{n} \left( \frac{1}{s - \lambda_i} \right) y_i^H B y_i(s)
\]

\(0\) if the \(i\)th mode is not controllable.

\[
y(s) = \sum_{i=1}^{n} \left( \frac{1}{s - \lambda_i} \right) (y_i^H C x_i) + \sum_{i=1}^{n} \left( \frac{1}{s - \lambda_i} \right) (C x_i)(y_i^H B) y_i(s) + D y_i(s)
\]

\(0\) if the \(i\)th mode is unobservable.

---

MIMO Residue Expansion

\[
y(s) = G(s) u(s) + f(s)
\]

\[
G(s) = \left\{ \sum_{i=1}^{n} \left( \frac{1}{s - \lambda_i} \right) C x_i y_i^H B \right\} + D
\]

\(R_i \equiv C x_i y_i^H B = \text{ith residue matrix at the pole } s = \lambda_i\)

If the \(i\)th mode is either uncontrollable or unobservable or both then \(R_i = 0\)

---

\(\)
Poles of \((A, B, C, D)\):

Poles are the eigenvalues of \(A\)

\[
x(t) = e^{At} = \left[ \sum_{i=1}^{n} e^{\lambda_i t} \right] \underbrace{y_i y_i^H}_{\text{scalar}}
\]

\(\gamma_i y_i^H\) is the degree that the \(i\)th initial condition excites the \(i\)th mode \(e^{\gamma_i t} x_i\)

Stable mode: If \(\gamma_i < 0\)

Unstable mode: If \(\gamma_i > 0\)

Stabilizability - Controllability

Let's consider the forced system,

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = 0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y &= Cx + Du
\end{align*}
\]

\[
\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j(t)
\]

\[
x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau
\]

\[
B e^{A(t)} \sum_{j=1}^{m} b_j u_j(t)
\]

\[
e^{A(t-t)} = \sum_{i=1}^{n} \lambda_i^{(t-t)} x_i y_i^H
\]

Therefore,
\[
X(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i(y_i^H b_j) \int e^{\lambda_i(t-t)} w_j(t) dt
\]

The scalar \((y_i^H b_j)\) can be interpreted as the degree that the \(j\)th control influences the \(i\)th mode.

If \((y_i^H b_j)\) is zero then the \(i\)th mode is uncontrollable from the \(j\)th input.

If any one of the modes is uncontrollable from all inputs,

\[
(y_i^H b_j) = \cdots = (y_i^H b_m) = 0 \quad \text{then,}
\]

\((A, B)\) is uncontrollable.

\[
\text{rank } [B, AB, \ldots, A^{n-1}B] < n.
\]

Definition: If all uncontrollable modes are stable then \((A, B)\) is stabilizable.

\[
\chi(s) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i(y_i^H b_j) \frac{1}{s - \lambda_i} w_j(s).
\]

or,

\[
\chi(s) = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} x_i y_i^H B w(s) \quad \text{and}
\]

\[
y(s) = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} (C x_i)(y_i^H B) w(s) + D w(s)
\]

MIMO Residue Expansion:
References on Multivariable Zeros


Before we discuss the multivariable zeros it is helpful to define the generalized eigenvalue problem. As it will be seen later, the solution to this problem provides an algorithm for calculating the transmission zeros.

The Generalized Eigenvalue Problem

In a generalized eigenvalue problem we deal with two \( m \times n \) matrices \( L \) and \( M \). Then the generalized eigenvalues \( \lambda_i \), right eigenvectors \( \mathbf{w}_i \), and left eigenvectors \( \mathbf{u}_i \) are defined by

\[
(s_i M - L) \mathbf{u}_i = 0 \quad ; \quad i = 1, \ldots, p \quad \text{(right)}
\]

\[
\mathbf{w}_i^H (s_i M - L) = 0 \quad ; \quad i = 1, \ldots, p \quad \text{(left)}
\]

Note that if \( A^{-1} \) exists, the generalized eigenvalue problem reduces to an ordinary eigenvalue problem and \( p = n \). If however, \( M \) is a singular matrix, then,

\[
0 \leq p < n
\]

The generalized eigenvalues \( s_i \) are still the roots of

\[
\text{det}(sM - L) = 0
\]

Let us now consider an LTI \( n \)-order multivariable system with \( m \) inputs and \( n \) outputs, given by:
\[
\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n
\]
\[
y = Cx, \quad y \in \mathbb{R}^m, \quad u \in \mathbb{R}^m.
\]

Then the system transfer matrix,
\[
G(s) = C(sI - A)^{-1}B
\]

and as we said before the poles \( \phi_i, i = 1, \ldots, n \) of \( G(s) \) are the solution of the ordinary eigenvalue problem
\[
(\rho; I - A)\psi_i = 0, \quad i = 1, \ldots, n
\]

The transmission zeros, \( \zeta_k, k = 1, \ldots, p \), span \( \mathbb{F} \) of \( G(s) \) are the solution to the generalized eigenvalue problem
\[
\begin{bmatrix}
2kI - A & -B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
x_k \\
\psi_k
\end{bmatrix} = 0, \quad k = 1, \ldots, p.
\]

where the \((nm)\) vector \( \begin{bmatrix} x_k \\ \psi_k \end{bmatrix} \) is the right generalized eigenvector corresponding to the generalized eigenvalue \( 2k \).

Using the above definitions, we can readily complete the relationship that defines the transmission zeros.
From the above matrix equation we have that:

\[
\det \begin{bmatrix}
2z_kI - A & -B \\
0 & 0
\end{bmatrix} = \det (z_kI - A) \det \epsilon (z_kI - A)^{-1}B = 0
\]

If we do not have a pole-zero cancellation then,

\[
\det (z_kI - A) \neq 0
\]

Thus, in the absence of pole-zero cancellation we must have:

\[
\det \epsilon (z_kI - A)^{-1}B = 0.
\]

However,

\[
G(s) = \epsilon (sI - A)^{-1}B.
\]

So,

\[
\det G(z_k) = 0
\]
Time Domain Interpretation of Transmission

Zeros.

In analogy to the definition of zeros in SISO systems, we would like to preserve the intuitive idea that the zeros of a system absorb the energy of a complex exponential signal. Thus, if we have a transmission zero at \( s = z_k \) and if the time-domain input vector \( u(t) \) has the form

\[
u(t) = u_0 e^{zt}
\]

where \( u_0 \) is a vector (a direction), then we want the output vector \( y(t) \) not to contain any signals of the form \( e^{zt} \). Note that in general, the output \( y(t) \) will contain terms of the form \( e^{p_i t} \), \( i = 1, \ldots, m \), where \( p_i \) are the poles, because the above input will excite the natural frequencies (or modes) of the system.

The above input-absorbing frequency concept must be refined a bit. This refinement involves the direction \( u_0 \) of the input.

Let's look at a two-input/two-output decoupled system with the transfer matrix
\[
G(s) = \begin{bmatrix}
\frac{s+1}{s^2} & 0 \\
0 & \frac{s+2}{s^2}
\end{bmatrix}
\]

For this system we should have two well-defined zeros at \( s = -1 \) and \( s = -2 \).

Now, if we apply a general input vector of the form
\[
\mu(t) = u_0 e^{-t}
\]
to the system, we can conclude that the output \( y_1(t) \) will not contain a term \( e^{-t} \), but that the output \( y_2(t) \) will. So, in a multivariable sense, the 2-D output vector \( y(t) \) will contain the zero-frequency exponential term \( e^{-t} \).

Similarly, if we apply an input vector of the form
\[
\mu(t) = u_0 e^{-2t}
\]
to the system then the output \( y_2(t) \) will not contain the frequency \( e^{-2t} \) but the output \( y_1(t) \) will.

The above argument indicates that we cannot use an arbitrary direction \( u_0 \) in the input. This simple example suggests that we retain the energy-absorbing property by a zero of an appropriate
input frequency by associating a specific direction of an input vector to each zero. In the example, if we consider the inputs,

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} e^{-t} \quad \text{and}
\]

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix} e^{-2t}
\]

we can see that the output vector \( y(t) \) will not contain the zero frequencies.

Thus, we are led to the multivariable concept that if a system \( G(s) \) has a transmission zero at \( s = 2 \omega \), and if we apply an input vector,

\[
y(t) = x_k e^{2\omega t}
\]

where \( x_k \) is an appropriate input direction associated with the transmission zero at \( s = 2 \omega \), then the output vector \( y(t) \) will not contain the (complex) exponential \( e^{2\omega t} \) in any of its components.

The above concept is still somewhat unsatisfactory from an analytical point of view. Even if the system is initially at rest, more precisely \( x(0) = 0 \), the output in response to the above input will contain, in general, terms of the
form \( e^{pt} \), where \( p_i \) are the system poles (eigenvalues of \( A \)). With a bit of thought it is obvious that it should be possible to select a specific initial state \( x(0) = x_k \) which will cause the output \( y(t) \) to remain identically zero. In other words, this selection of the initial state wipes out the natural dependence of the output upon the pole frequencies.

We then arrive at the following interpretation of a multivariable system transmission zero.

**Definition.** Given a system, we say that the system has a (complex) zero at \( s = \frac{1}{\lambda_k} \) if there exists a vector \( u_k \in \mathbb{R}^m \) and a vector \( x_k \) in \( \mathbb{R}^n \) with the following properties:

1. If the initial system state is \( x(0) = x_k \) and
2. If the input vector \( u(t) \) is given by
   \[ u(t) = u_ke^{\lambda_k t}, \quad t \geq 0 \]

then the output \( y(t) \) is identically zero, i.e.,
\[ y(t) = 0 \quad \text{for all} \quad t \geq 0. \]
Now we need to demonstrate that the solution of the generalized eigenvalue problem, as defined previously, does indeed satisfy the requirements of the above definition.

First note that the generalized eigenvalue problem implies two relations:

\[(A - \lambda I)\mathbf{x}_k - \beta \mathbf{y}_k = 0\]
\[C \mathbf{x}_k = 0.\]

By taking the \(L\)-transform of the state equation:
\[
\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)
\]
with a non-zero initial condition,
\[
S\mathbf{x}(s) - \mathbf{x}(0) = A\mathbf{x}(s) + B\mathbf{u}(s)
\]

If the input \(\mathbf{u}(t) = \mathbf{y}_k e^{\lambda t}\), then its \(L\)-transform is:
\[
\mathbf{y}(s) = \frac{1}{s - \lambda} \mathbf{y}_k
\]

Also, assume that the initial state \(\mathbf{x}(0)\) is selected such as:
\[
\mathbf{x}(0) = \mathbf{x}_k
\]

where \(\mathbf{z}_k, \mathbf{x}_k\) and \(\mathbf{y}_k\) satisfy the two relations obtained by the generalized eigenvalue problem.
Substituting $z_k(t)$ and $x(0)$ into the state equation,

$$(sI-A)x(s) = x(0) + B u(s) =$$

$$= x_k + \frac{1}{s-z_k} B u_k =$$

$$= \frac{1}{s-z_k} \left[ s x_k - z_k x_k + B u_k \right]$$

$$= \frac{1}{s-z_k} \left[ s x_k - z_k x_k + A x_k x_k + B u_k \right]$$

$$= \frac{1}{s-z_k} \left[ (sI-A)x_k - (z_k I - A)x_k + B u_k \right]$$

In view of the specific relationships given by the generalized eigenvalue problem,

$$(sI-A)x(s) = (sI-A) \frac{1}{s-z_k} x_k$$

Hence,

$$x(s) = \frac{1}{s-z_k} x_k$$

By taking the $L^1$,

$$x(t) = e^{Z_k t} x_k$$

which implies that the specific way that the directions $x_k$ and $u_k$ have been selected according to the first relation of the generalized eigenvalue problem,
the dependence of the state $y(t)$ on the pole frequencies has been eliminated.

The output of $y(t)$ is given by, in view of the second relation of the generalized eigenvalue problem, as:

$$y(t) = Cx(t) = e^{\lambda t} \leq x_k = 0 \quad \text{for all } t > 0$$

which establishes the desired result.

The above results stated by Kailath [5, pp. 446-451] w/o proof; Kailath refers to the paper by Beroz & Seshulman [3] in which a pure transfer function argument is used. However, it is assumed that the system starts from zero state and in order to have zero output it is necessary to add to the input $u(t) = y_k e^{\lambda t}$ a bunch of impulses, and higher order singularity functions. In other words, the introduction of these singularity functions is to cause the state at some $t = 0^+$ to be $x(0^+) = x_k$. 
Consider an \( n \)-th order MIMO system,

\[
\begin{align*}
\dot{x}(s) &= A x(s) + B e(s) \\
y(s) &= C x(s)
\end{align*}
\]

Assume that \( e(s), x(s) \), and \( y(s) \) are \( n \)-dimensional vectors and that \( G(s) \) is the loop transfer matrix with any pole-zero cancellations. \( G(s) \) is an \( n \times n \) complex matrix. Note that \( G(s) \) includes the open-loop dynamics and the dynamics and gains of any compensation in the forward loop.

Assume that \( G(s) \) is given in the time-domain by:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B e(t) \\
y(t) &= C x(t)
\end{align*}
\]

where, \( x(t) \) is the \( n \)-dimensional state vector, \( A \) is \( n \times n \) matrix, \( B \) is \( n \times m \), and \( C \) is \( m \times n \).

It is known that:

\[
G(s) = C (s I - A)^{-1} B
\]

The eigenvalues of the \( A \) matrix are the poles of \( G(s) \) and are the open-loop poles of the LTI system \( G(s) \).
Definition: The open-loop characteristic polynomial, denoted by $\phi_0(s)$, is defined by:

$$\phi_0(s) = \det(sI-A).$$

The open-loop poles are the roots of $\phi_0(s)$, since these are identical to the eigenvalues of $A$.

The following relation holds:

$$y(s) = G(s) G_0(s)$$

where

$$G_0(s) = [I + G(s)]^{-1}$$

$m \times m$ closed-loop transfer matrix.

Proof: Let $A$ be a square matrix, then,

\[
\begin{align*}
A + A^2 &= A + A^2 \\
A(I + A) &= (I + A)A \\
(I + A)^{-1} A (I + A) &= A \\
(I + A)^{-1} A &= A (I + A)^{-1}
\end{align*}
\]

provided that the inverse exists.

We are now interested in the poles of the closed-loop system. It is easy to see that:

$$\dot{x}(t) = [A - BC]x(t) + BU(t)$$

$$y(t) = Cx(t)$$
The stability of the closed-loop system will depend upon the eigenvalues of the closed-loop system matrix, $A_{cl}$, i.e.,

$$A_{cl} = A - BC$$

**Definition:** The closed-loop characteristic polynomial, $\phi_{cl}(s)$, is defined by

$$\phi_{cl}(s) = \det(sI - A + BC) = \det(sI - A_{cl})$$

The roots of this poly are the poles of the closed-loop system.

Now, it is easy to show that:

$$\bar{G}_{cl}(s) = C(sI - A + BC)^{-1}B$$

However, (using a previously mentioned relation)

$$\left\{ C(sI - A)^{-1}B \left[ I + C(sI - A)^{-1}B \right]^{-1} \right\}^{-1} = C(sI - A + BC)^{-1}B$$

Now, it is relatively not so easy to show that,

$$\det(sI - A + BC) = \det(sI - A) \cdot \det[I + C(sI - A)^{-1}B]$$

or equivalently,

$$\phi_{cl}(s) = \phi_{cl}(s) \cdot \det[I + C(sI - A)^{-1}B]$$
SOME USEFUL SINGULAR VALUE INEQUALITIES

As we shall see a bit later there are some quantities that are appearing repeatedly and are very important in the robustness results of MIMO systems. So, we will try to give some inequalities involving the SV of some important quantities. Such as:

\[
\min (I + G(j\omega)) \quad \text{and} \quad \min (I + G^{-1}(j\omega))
\]

where \(G(j\omega)\) is the LTM.

First we prove that:

\[
(I + G)^{-1} + (I + G^{-1})^{-1} = I \tag{1}
\]

Proof:

\[
I + G = G + I \quad I + G^{-1} = G(I + G^{-1}) \quad (I + G)(I + G^{-1})^{-1} = G
\]

\[
I + (I + G)(I + G^{-1})^{-1} = I + G \quad (I + G)(I + G^{-1})^{-1} = I + G
\]

\[
(I + G)^{-1} + (I + G^{-1})^{-1} = I
\]

\[
(I + G)^{-1} + \left(\frac{I + G}{I}\right)(I + G^{-1})^{-1} = I
\]

\[
(I + G)^{-1} + (I + G^{-1})^{-1} = I
\]
In addition, we have that
\[ \sigma_{\min}(A^t) = \sigma_{\max}(A^{-1}) \]  
which is the result of the min of max. SV's of a matrix A. Using (1) and (2) we can prove the following:

\[ \sigma_{\min}^{-1}(I + S) + \sigma_{\min}^{-1}(I + S^{-1}) \geq 1 \]  
\[ \sigma_{\min}^{-1}(I + S) + 1 \geq \sigma_{\min}^{-1}(I + S^{-1}) \]  
\[ \sigma_{\min}^{-1}(I + S^{-1}) + 1 \geq \sigma_{\min}^{-1}(I + S) \]  
\[ \sigma_{\max}(S) \geq \frac{\sigma_{\min}(I + S)}{\sigma_{\min}(I + S^{-1})} \geq \sigma_{\min}(S) \]

**Proof of (4)**

Let denote \( \| \cdot \|_2 \) the spectral norm defined by:

\[ \sigma_{\max}(A) = \| A \|_2 \]
\[ \sigma_{\min}(A) = \| A^{-1} \|_2 \]

So,
\[ \sigma_{\min}^{-1}(A) = \| A^{-1} \|_2 \]

Now, take the norm of both sides of eq. (1):
\[ 1 = \| (I + S)^{-1} + (I + S^{-1}) \|_2 \]
\[ \leq \| (I + S)^{-1} \|_2 + \| (I + S^{-1}) \|_2 \]
\[ = \sigma_{\min}^{-1}(I + S) + \sigma_{\min}^{-1}(I + S^{-1}) \]
Proof of (4):

From (1) \[ \Rightarrow \quad (I + G^{-1})^{-1} = I - (I + G)^{-1} \]

Take the norms:

\[ \| (I + G^{-1})^{-1} \|_2 = \| I - (I + G)^{-1} \|_2 \]

\[ \leq \| I \|_2 + \| (I + G)^{-1} \|_2 \]

Hence,

\[ \sigma_{\text{min}} (I + G^{-1}) \leq 1 + \sigma_{\text{min}}^{-1} (I + G) \]

Proof of (5) & (6): Left as an exercise.
JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
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- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
- LQR, KF, LQG
- LQ-Servo, LQG/LTR
Uncertainty Modeling

- MINIMIZING THE EFFECTS OF UNCERTAINTY VIA FEEDBACK
- TWO MODELS FOR TREATING UNCERTAINTY
- UNSTRUCTURED UNCERTAINTY
- PUTTING THINGS IN STANDARD FORM
**Uncertainty Modeling (Cont'd)**

**FOR PERFECTLY KNOWN PLANTS:**

\[ r(s) \xrightarrow{G_d(s)} G^{-1}(s) \xrightarrow{G(s)} y(s) = G_d(s) r(s) \]

**FOR UNCERTAIN PLANTS:**

\[ r(s) \xrightarrow{G_d(s)} G^{-1}(s) \xrightarrow{\text{Real Plant}} y(s) \neq G_d(s) r(s) \]
Uncertainty Modeling (Cont'd)

- **FEEDBACK REDUCES SENSITIVITY TO MODELING UNCERTAINTY:**

\[ y(s) \approx G_d(s) r(s) \]

- **UNDER OPEN-LOOP CONTROL A 10 % VARIATION IN THE PLANT MODEL WILL TRANSLATE TO 10 % VARIATION IN \( y(s) \).**

- **UNDER CLOSED-LOOP CONTROL IF THE RETURN DIFFERENCE IS KEPT AT APPROXIMATELY 100 THEN A 10 % VARIATION IN THE PLANT WILL TRANSLATE TO ABOUT 0.1 % VARIATION IN \( y(s) \).**
Uncertainty Modeling (Cont'd)

- Modeling uncertainties are ever present.

- Reasons:
  - Incorrect parameter values,
  - Environment noise,
  - Unmodeled dynamics (actuators, sensors, resonances),
  - Time delays,
  - Unmodeled nonlinearities,
  - Space-dependent effects.

- Modeling uncertainties usually small at low frequencies and increase as the frequency increases.

- Therefore, it is important to model uncertainty.
Uncertainty Modeling (Cont'd)

TWO BROAD CATEGORIES OF UNCERTAINTIES

- STRUCTURED UNCERTAINTY (PARAMETRIC)
  
  MODELING ERRORS WHICH CAN BE ELIMINATED BY ADJUSTING PARAMETER VALUES TO BETTER MATCH THE DATA.

- UNSTRUCTURED UNCERTAINTY (NON-PARAMETRIC)
  
  ALL OF THE OTHER TYPES OF MODELING ERRORS.

Note: NON-PARAMETRIC STRUCTURED UNCERTAINTY.
Uncertainty Modeling (Cont'd)
Unstructured Uncertainty

If \( G_0(s) \) is the nominal model, then we can assume the following unstructured uncertainty models:

\[
G(s) = G_0(s) + \Delta_\alpha(s) \quad \text{--- additive perturbation}
\]

\[
\underline{G}(s) = G_0(s) \left[ I + \Delta_\Gamma(s) \right] \quad \text{or} \quad \text{Input multiplicative perturbation}
\]

\[
\overline{G}(s) = \left[ I + \Delta_\beta(s) \right] G_0(s) \quad \text{Output multiplicative perturbation}
\]

The size of the perturbations is measured by \( \|\cdot\|_\infty \).

\( \Delta_\alpha(s) \) or \( \Delta_\beta(s) \) are more realistic as compared to \( \Delta_\Gamma(s) \) because:

\( \|\Delta_\Gamma(s)\|_\infty \leq 0.15 \) means that the size of the perturbation is at most 15% of \( G_0(s) \), since,

\[
\| G(s) - G_0(s) \|_\infty = \| G_0(s) \Delta_\Gamma(s) \|_\infty \leq \| G_0(s) \|_\infty \cdot \| \Delta_\Gamma(s) \|_\infty \\leq 0.15 \| G_0(s) \|_\infty .
\]

whereas, \( \|\Delta_\beta(s)\|_\infty \leq 0.15 \) means,

\[
\| G(s) - G_0(s) \|_\infty = \| \Delta_\beta(s) \|_\infty \leq 0.1 \]
Uncertainty Modeling (Cont'd)

UNSTRUCTURED UNCERTAINTY

- ADDITIVE NORM-BOUNDED

\[ \text{Error} = (\text{Model} + \text{Norm-Bounded Uncertainty} (\Delta)) - \text{(Model)} \]

(Some sort of absolute error measure)

- MULTIPLICATIVE NORM-BOUNDED

\[ \text{Error} = (1 + \text{Norm-Bounded Uncertainty} (\Delta)) \text{ Model} - \text{(Model)} \]

(Some sort of relative error measure)

Note: Multiplicative error more realistic.
Uncertainty Modeling (Cont'd)

- General Form

\[ z \rightarrow \Delta \rightarrow d \]

\[ \| \Delta / W \|_{H^\infty} \leq 1 \]

Bode Diagram of \( \Delta \):
- Log mag: 0
- Phase: -360

W

Log w
Uncertainty Modeling (Cont'd)

- Normalized Form

\[ z \xrightarrow{\Delta'} W \xrightarrow{d} \quad \| \Delta' \|_{H^\infty} \leq 1 \]

Bode Diagram of $\Delta'$

- Log magnitude
  - 0
  - 0
  - log w

- Phase
  - 0
  - -360
Assume that we have the following unstructured uncertainty:

\[ \Delta(s) \]

The Bode diagram of \( \Delta(s) \) would look like:

\[ \frac{\Delta(s)}{W(s)} \]

So, we could express the unstructured uncertainty as:

\[ \| \frac{\Delta(s)}{W(s)} \|_{\infty} < 1 \]

If I rename this, \( \Delta'(s) = \frac{\Delta(s)}{W(s)} \)

then, \( \| \Delta'(s) \|_{\infty} < 1 \) and
Uncertainty Modeling (Cont'd)

STANDARD FORM OF DESCRIBING FEEDBACK LOOPS
(THE FOLLOWING TWO BLOCK DIAGRAMS ARE EQUIVALENT)
EXAMPLE:

\[ v \rightarrow e \rightarrow K \rightarrow u \rightarrow G_0 \rightarrow y \]

\[ z \rightarrow \Delta_a \]

\[ v \rightarrow e \rightarrow K \rightarrow W_1 \rightarrow z \rightarrow G_0 \rightarrow u \rightarrow W_2 \rightarrow \hat{u} \]

\[ v \rightarrow z \rightarrow W_1 \rightarrow G_0 \rightarrow u \rightarrow W_2 \rightarrow \hat{u} \]
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Issues in Multivariable Control Systems

- BASIC MIMO FEEDBACK STRUCTURE
- OPEN-LOOP AND CLOSED-LOOP TRANSFER FUNCTION MATRICES
- NOMINAL STABILITY
- NOMINAL PERFORMANCE
- TWO IMPORTANT TOOLS: GENERALIZED NYQUIST & SMALL-GAIN THEOREM
- ROBUST STABILITY
- ROBUST PERFORMANCE

(Example)
 Issues in Multivariable Control Systems (Cont'd)
TRANSFER FUNCTION MATRIX EXPRESSIONS

\[ y(s) = \left[ I + G(s)K(s) \right]^{-1} G(s)K(s)r(s) \]

\[ + \left[ I + G(s)K(s) \right]^{-1} d_0(s) \]

\[ + \left[ I + G(s)K(s) \right]^{-1} G(s)d_1(s) \]

\[ - \left[ I + G(s)K(s) \right]^{-1} G(s)K(s)n(s) \]

\[ G(s)K(s) : \text{Forward Loop Transfer Function Matrix} \]

\[ \left[ I + G(s)K(s) \right] : \text{Return Difference Transfer Function Matrix} \]

\[ \left[ I + G(s)K(s) \right]^{-1} : \text{Sensitivity Transfer Function Matrix \,(S(s))} \]

\[ \left[ I + G(s)K(s) \right]^{-1} G(s)K(s) : \text{Complementary Sensitivity TFM \,(T(s))} \]

\[ \text{Closed -Loop TFM} \]

\[ \text{NOTE : } S(s) + T(s) = I \]
THE FOUR BASIC ISSUES IN THE ANALYSIS AND DESIGN OF FEEDBACK CONTROL SYSTEMS

- NOMINAL STABILITY
- NOMINAL PERFORMANCE
- ROBUST STABILITY
- ROBUST PERFORMANCE
Issues in Multivariable Control Systems (Cont'd)

- NOMINAL STABILITY
  "DETERMINED VIA EIGENVALUE TESTING"
  NOTE: INHERENT IN MOST MIMO DESIGN METHODS.

- NOMINAL PERFORMANCE
  - COMMAND FOLLOWING
  - DISTURBANCE REJECTION
  - SENSITIVITY MINIMIZATION
  - SENSOR NOISE RESPONSE
Issues in Multivariable Control Systems (Cont'd)

- COMMAND FOLLOWING
  "LARGE LOOP TFM G(s) K(s) IN THE FREQUENCY RANGE OF INTEREST"

- DISTURBANCE REJECTION
  "LARGE LOOP TFM G(s) K(s) IN THE FREQUENCY RANGE OF INTEREST"

- SENSITIVITY REDUCTION
  "SMALL SENSITIVITY TFM OR LARGE LOOP TFM G(s) K(s) IN THE FREQUENCY RANGE OF INTEREST"

- SENSOR NOISE REDUCTION
  "SMALL CLOSED-LOOP TFM OR SMALL LOOP TFM G(s) K(s) IN THE FREQUENCY RANGE OF INTEREST"
Issues in Multivariable Control Systems (Cont'd)

MIMO LOOP SHAPING USING FORWARD LOOP SINGULAR VALUES

- GOOD COMMAND FOLLOWING & DISTURBANCE REJECTION

"LARGE FORWARD LOOP GAIN IN THE FREQUENCY RANGE OF INTEREST"

- SENSOR NOISE REJECTION

"SMALL FORWARD LOOP GAIN IN THE FREQUENCY RANGE OF INTEREST"
Issues in Multivariable Control Systems (Cont'd)
MIMO LOOP SHAPING USING SENSITIVITY SINGULAR VALUES

\[
\max_{\omega} \left( \sigma_{\max} \left[ w_p^{-1}(j\omega) \left( I + G(j\omega) K(j\omega) \right)^{-1} \right] \right) < 1
\]
Issues in Multivariable Control Systems (Cont'd)

- TWO IMPORTANT TOOLS FOR ROBUSTNESS ANALYSIS
- GENERALIZED NYQUIST CRITERION
- "EXTENSION OF THE SISO TOOL IN THE MIMO DOMAIN"
- SMALL-GAIN THEOREM
- "A FEEDBACK LOOP COMPOSED OF STABLE TFMs WILL REMAIN STABLE IF THE PRODUCT OF ALL TFMs GAINS REMAINS SMALLER THAN UNITY"
INTRODUCTION TO CONTROL SYSTEM ROBUSTNESS:

We have seen that:
\[ \phi_{cl}(s) = \phi_{ol}(s) \cdot \det[I + G(s)] \]

Now the open-loop system can have unstable and stable zeros.

MIMO

Nyquist Theorem:

The closed-loop system is stable i.e. \( \phi_{cl}(s) \) has no roots in the RHP. i.e. and only if
\[ \text{N}(0, \det[I + G(s)], D_R) = -P_u \]

\( \text{N}(0, \det[I + G(s)], D_R) \) = -P_u

The # of clockwise encirclements of the point 0 by the clockwise contour of the map \( \det[I + G(s)] \) under the map \( \det[I + G(s)] \), D_R = -P_u.

\( \text{N}(0, \det[I + G(s)], D_R) \) = -P_u

\( \text{N}(0, \det[I + G(s)], D_R) \) = -P_u

Note: Distance from critical point in SISO means good performance, but how about MIMO system?

i.e. Suppose that \( \det[I + G(s)] \) is "large"
\[ \Rightarrow \text{gain} [I + G(s)] \text{ is "large"} \]
Conclusions: Distances in the micro Nyquist diagram do not necessarily translate directly into singular value information.

Define a nominal plant, \( G(s) \).

\[ \begin{array}{c}
V(s) + \\
\downarrow \\
K(s) \\
\downarrow \\
G(s) \\
\downarrow \\
\uparrow \\
\text{Cloud} \\
\text{Nominal loop is stable,}
\end{array} \]

Nominal LFM, \( T(s) = G(s) \cdot K(s) \).

But, models have limitations (\ldots)

Therefore, if actual plant is \( \tilde{G}(s) \) then,

\[ \tilde{G}(s) = L(s) \cdot G(s) \]

\[ \begin{array}{c}
\tilde{G}(s) \\
\downarrow \\
\text{Cloud} \\
\text{For nominal case,} \quad L(s) = I
\end{array} \]
\( \sigma_{\min} [1 + G(s)K(s)] > r \) means that the Nyquist plot stays farther than \( r \) from the critical point at \(-1\).

\[
GM = \text{gain margin} = \frac{1}{1 - r}
\]

\[
PM = \text{phase margin} = 2 \sin^{-1}\left(\frac{r}{2}\right)
\]

Ex. \( r = 0.66 \quad \Rightarrow \quad GM = 3 \quad \& \quad PM = 38^\circ \)
Basic Problem:

Robustness tests for MIMO feedback control systems hinge on the existence of modeling errors that change the number of encirclements of the critical point in the Nyquist diagram as compared to those of the nominal design.

In particular, there can be a modeling error that causes the inverse difference matrix of the actual system to become singular, thereby just "touching" the critical point, so that the number of encirclements is ready to change.

So, some singular value inequalities of arbitrary complex matrices are presented, which can be used in conjunction with the MIMO Nyquist criterion to establish the stability robustness of a given MIMO design.

Let \( \tilde{A} \) be an \( n \times n \) complex matrix. Assume that it is nonsingular so that \( \tilde{A}^{-1} \) exists. Now let \( \tilde{\tilde{A}} \) be another nonsingular \( n \times n \) complex matrix. Think of \( \tilde{\tilde{A}} \) as a "model" of \( \tilde{A} \) and later on we'll define several "error" structures that can relate \( \tilde{A} \) and \( \tilde{\tilde{A}} \).

The basic questions that we'll be asking are related to whether the matrix \( (\tilde{I} + \tilde{F}) \) is singular or not. Also, we'll be looking at sufficient
Conditions that involve singular values which will guarantee that \((\mathbf{I} + \mathbf{A})\) is nonsingular.

There are several ways that we can express the difference between \(\mathbf{A} \neq \mathbf{A}^\prime\). The idea is to define an error matrix such that:

\[
\text{when } E = 0 \text{ then } \mathbf{A}^\prime = \mathbf{A}.
\]

1. **Additive Error Matrix**

\[
E = \mathbf{A}^\prime - \mathbf{A} \quad \text{or} \quad \mathbf{A}^\prime = \mathbf{A} + E
\]

2. **Subtractive Error Matrix**

Assuming both \(\mathbf{A}^{-1} \neq \mathbf{A}^\prime^{-1}\) exist:

\[
E = \mathbf{A}^{-1} - \mathbf{A}^{-1} \quad \text{which is equivalent to}
\]

\[
\mathbf{A}^\prime = (\mathbf{A}^{-1} + E)^{-1}
\]

3. **Post-Multiplicative Error Matrix**

In this case:

\[
E = \mathbf{A}^{-1} (\mathbf{A}^\prime - \mathbf{A}) \quad \text{or equivalently,}
\]

\[
\mathbf{A}^\prime = \mathbf{A} (\mathbf{I} + E)
\]
4. **Pre-Multiplicative Error Matrix**

Now the error matrix is:

\[ E = (\tilde{A} - A)A^{-1} \quad \text{or,} \]

\[ \tilde{A} = (I + E)A \]

5. **Post-division error matrix**

In this formulation, the error matrix is defined by:

\[ E = (\tilde{A}^{-1} - A^{-1})A \quad \text{& assuming the inverses exist:} \]

\[ \tilde{A} = A(I + E)^{-1} \]

6. **Pre-division error matrix**

In this category, define:

\[ E = A(\tilde{A}^{-1} - A^{-1}) \quad \text{or,} \]

\[ \tilde{A} = (I + E)^{-1}A \]

**Discussion:** The additive & subtractive error matrices, are often referred to as absolute error matrices. The multiplicative and division error matrices are referred to as relative error matrices.
To further understand the differences between absolute & relative error matrices consider the following situation:

\[
\begin{align*}
A &= XY \\
\tilde{A} &= \bar{X} \tilde{Y}
\end{align*}
\]

Let \( E_Y \) denote the additive error between \( Y \) and \( \tilde{Y} \) and let \( E_A \) be the additive error between \( \tilde{A} \) and \( \bar{A} \). The proper way of thinking is that \( X \) is a model of \( \bar{Y} \), that there is no error in \( \bar{X} \) & that \( \tilde{A} \) is a model of \( \bar{A} \). Thus, the additive error matrices are obviously related by

\[
E_A = \tilde{A} - \bar{A} = \bar{X} (\tilde{Y} - \bar{Y}) = \bar{X} E_Y
\]

The additive error \( E_Y \) is changed by the matrix \( \bar{X} \) to generate \( E_A \). Clearly, the additive error matrices are not equal,

\[
E_A \neq E_Y
\]

One would arrive at the same conclusion if one used the subtractive error matrix.

Now, let’s use the “right” relative error matrix. As before, let

\[
A = XY \quad \tilde{A} = \bar{X} \tilde{Y}
\]

Let,

\[
E_Y \text{ be the post-multiplicative error, i.e.}
\]

\[
E_Y = Y^{-1} (\tilde{Y} - Y)
\]

\[
E_A = A^{-1} (\tilde{A} - \bar{A})
\]
We are going to demonstrate that $E_x = E_y$ independent of $x$.

\[ E_x = A^{-1} (\tilde{A} - A) = (XY)^{-1} (X\tilde{X} - XY) = \]
\[ = Y^{-1} X^{-1} X (Y - Y) = \]
\[ = Y^{-1} (Y - Y) = E_y \]

**Problem 1:** Given that $(I + \tilde{A})$ is non-singular, determine a sufficient condition that $(I + \tilde{A})$ is also non-singular.

**Use of Additive Error Matrix**

Suppose that $(I + \tilde{A})$ is singular. Let $E$ denote the additive error matrix.

\[ E = \tilde{A} - A \]

Hence, $(I + A + E)$ is singular. Therefore, there exists a nonzero vector $x$ in the null space of $(I + A + E)$ such that

\[ (I + A + E)x = 0 ; \ x \neq 0 \]

It follows that:

\[ (I + A)x = -Ex ; \ x \neq 0 \]

$x \in N$ denotes the fact that $x$ belongs to the null space of $(I + A + E)$. Note that this implies that $x$ is constrained, i.e., cannot be an arbitrary vector.
Now, 

\[ \| (I + A) x \|_2 = \| E x \|_2 ; x \in \mathbb{R} \]

and, 

\[ \frac{\| (I + A) x \|_2}{\| x \|_2} = \frac{\| E x \|_2}{\| x \|_2} ; x \in \mathbb{R} \]

By the definition of singular values, 

\[ \min \left( \frac{\| (I + A) y \|_2}{\| y \|_2} \right) \]

\[ \leq \min_{x \in \mathbb{R}} \frac{\| (I + A) x \|_2}{\| x \|_2} \]

\[ \leq \min_{x \in \mathbb{R}} \frac{\| E x \|_2}{\| x \|_2} \]

\[ \leq \max_{y \neq 0} \frac{\| E y \|_2}{\| y \|_2} \]

\[ = 6 \max \left( E \right) \]

(*) This states that the result from an unconstrained minimization (left-hand side) must be smaller than the result of a constrained minimization (right-hand side) which results from the fact that \( y \) is not arbitrary but, rather, \( x \in \mathbb{R} \).

(**) Similarly, the result of an unconstrained
maximization (RHS) must be larger than the result of a constrained minimization (LHS).

Result: If \((I+\hat{A})\) is singular, then the above inequality must be true. Therefore, if

\[ \sigma_{\text{max}}(E) \leq \sigma_{\text{min}}(I+A) \]

then, \((I+\hat{A})\) must be nonsingular.

So, the inequality above where \(E = \hat{A} - A\) is a sufficient condition for the matrix \((I+\hat{A})\) to be nonsingular.

<table>
<thead>
<tr>
<th>Error Structure</th>
<th>Sufficient Condition for (\det(I+\hat{A}) \neq 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>(\sigma_{\text{max}}(E) \leq \sigma_{\text{min}}(I+A))</td>
</tr>
<tr>
<td>Subtractive</td>
<td>(\sigma_{\text{max}}(E) &lt; \sigma_{\text{min}}(I+A^{-1}))</td>
</tr>
<tr>
<td>Post-Multiply</td>
<td>(\sigma_{\text{max}}(E) &lt; \sigma_{\text{min}}(I+A^{-1}))</td>
</tr>
<tr>
<td>Pre-Multiply</td>
<td>(\sigma_{\text{max}}(E) &lt; \sigma_{\text{min}}(I+A^{-1}))</td>
</tr>
<tr>
<td>Post-Dir.</td>
<td>(\sigma_{\text{max}}(E) &lt; \sigma_{\text{min}}(I+A))</td>
</tr>
<tr>
<td>Pre-Dir.</td>
<td>(\sigma_{\text{max}}(E) &lt; \sigma_{\text{min}}(I+A))</td>
</tr>
</tbody>
</table>
**Stability Robustness**

"Is this loop stable?"

\[
\tilde{T}(s) = \tilde{G}(s) \cdot K(s) = L(s) G(s) K(s) = \left[ I + E \tilde{M}(s) \right] G(s) K(s)
\]

Note that, \( K(s) \) is not adaptive.

**Theorem:** Stability - robustness

**Assumptions:**
1. \( ||T(jw)|| \leq ||\tilde{T}(jw)|| \to 0 \text{ as } \omega \to \infty \)
2. The number of unstable poles of \( \tilde{T}(s) \) is the same as for \( \tilde{T}(s) \) (not necessarily in the same location!)
3. If \( \tilde{T}(s) \) has poles on \( j\omega \) axis, then \( \tilde{T}(s) \) should have the same poles on the \( j\omega \) axis.

If the nominal loop is stable, then the actual loop is stable if:

(A) \( \sigma_{\max} \left[ \tilde{L}(j\omega) - I \right] < \sigma_{\min} \left[ I + I(j\omega) \right] \)

or if (use for high freq.)

(B) \( \sigma_{\max} \left[ L(j\omega) - I \right] < \sigma_{\min} \left[ I + I(j\omega) \right] \)

Note: Only sufficient condition for stability of the
actual plant.

Define, $E_m(j\omega) \equiv \frac{1}{j\omega} - I$ then, (8) is:

$$\sigma_{\max} \left[ E_m(j\omega) \right] < \sigma_{\min} \left[ \xi(j\omega) \right] = \frac{1}{\sigma_{\max} \left[ \xi(j\omega) \right]}$$

or:

$$\sigma_{\max} \left[ \xi(j\omega) \right] < \sigma_{\min} \left[ E_m(j\omega) \right] \quad (8')$$

Where $\sigma_{\max} \left[ E_m(j\omega) \right]$ and $\sigma_{\min} \left[ E_m(j\omega) \right]$ are the maximum and minimum singular values of $E_m(j\omega)$, respectively.

Diagram:  

- $\sigma_{\max}$ and $\sigma_{\min}$ vs. $\log_{10}$.
- $E_m(j\omega)$ and $\xi(j\omega)$ curves.

Diagram labels:
- $E_m(j\omega)$
- $\xi(j\omega)$
- $\sigma_{\max}$
- $\sigma_{\min}$
- $\log_{10}$
The stability robustness result states that, if
\[ \sigma_{\text{max}} \left[ \begin{bmatrix} I + G(s)K(s) \end{bmatrix}^{-1} G(s)K(s) \right] < \sigma_{\text{min}} \left[ E_{\text{m}}(s) \right] \]
for \( w \in R \)
then we have robust stability for modeling errors reflected in \( E_{\text{m}}(s) \).

Let's put this in the context of unstructured uncertainty. If we name \( \Delta(s) = E_{\text{m}}(s) \) and assume that
\[ \sigma_{\text{max}} (\Delta(s)) \leq W_{\text{m}}(s) \Rightarrow \]
\[ \frac{1}{\sigma_{\text{min}} (\Delta^{-1}(s))} \leq W_{\text{m}}(s) \Rightarrow \]
\[ \sigma_{\text{min}} (\Delta^{-1}(s)) \geq \frac{1}{W_{\text{m}}(s)} \]

So, if \( \text{Comp. Sens. or Closed-loop TOV} \)
\[ \sigma_{\text{max}} \left[ T(s) \right] < \frac{1}{W_{\text{m}}(s)} \text{ then } \sigma_{\text{min}} (\Delta^{-1}(s)) \]
or we guarantee robust stability.

This is sometimes expressed as:
If nominal stability is guaranteed then,

\[ \sigma_{\max} \left\{ \left[ I + G(j\omega) K(j\omega) \right]^{-1} \right\} < \sigma_{\min} \left[ E^{-1}(j\omega) \right] \]

or

\[ \sigma_{\max} \left\{ \left[ I + G(j\omega) K(j\omega) \right]^{-1} G(j\omega) K(j\omega) \right\} < \frac{1}{W_m(j\omega)} \]

where,

\[ \sigma_{\max} \left[ \Delta(j\omega) \right] \leq W_m(j\omega) \]

or

\[ \max_{\omega} \left[ \sigma_{\max} \left\{ \left[ I + G(j\omega) K(j\omega) \right]^{-1} \right\} W_m(j\omega) \right] < 1 \]
\[
\left\| \left( I + \hat{G}(j\omega)K(j\omega) \right)^{-1} \hat{G}(j\omega)K(j\omega) W_{p}(j\omega) \right\|_{\infty} < 1
\]

or

\[
\sigma_{\max} \left[ I + \hat{G}(j\omega)K(j\omega) \right]^{-1} < 1
\]

Go back to **nominal performance**:

\[
\sigma_{\max} \left[ I + \hat{G}(j\omega)K(j\omega) \right]^{-1} < 1
\]

Define:

\[
W_{p}(s) > 1 \quad \text{and} \quad W_{p}^{-1}(s) < 1
\]

Then,

\[
\sigma_{\max} \left[ I + \hat{G}(j\omega)K(j\omega) \right]^{-1} < \frac{1}{W_{p}(j\omega)}
\]

or

\[
\max_{\omega} \left\{ \sigma_{\max} \left[ I + \hat{G}(j\omega)K(j\omega) \right]^{-1} W_{p}(j\omega) \right\} < 1
\]

or

\[
\left\| \sigma_{\max} \left( I + \hat{G}(j\omega)K(j\omega) \right)^{-1} W_{p}(j\omega) \right\|_{\infty} < 1
\]
This shaping is similar to the open-loop TMS shaping we did in the earlier stages of the course. But this is done in a closed-loop manner.
Issues in Multivariable Control Systems (Cont'd)

- ROBUST STABILITY
  CONSIDER ROBUST STABILITY UNDER MULTIPLICATIVE UNSTRUCTURED UNCERTAINTY

\[
\begin{align*}
\max_{\omega} \left( \sigma_{\max} \left[ \left(1 + G(j\omega)K(j\omega) \right)^{-1} \right] \right) &< 1 \\
\max_{\omega} \left[ \Delta(j\omega) \right] &< \sigma_{\max} \left[ W_m(j\omega) \right]
\end{align*}
\]

IF THE NOMINAL LOOP IS STABLE THEN ROBUST STABILITY IS POSSIBLE IF

- \( \max_{\omega} \left( \sigma_{\max} \left[ \left(1 + G(j\omega)K(j\omega) \right)^{-1} \right] \right) < 1 \)
- \( \max_{\omega} \left[ \Delta(j\omega) \right] < \sigma_{\max} \left[ W_m(j\omega) \right] \)

WHERE \( \sigma_{\max} \left[ \Delta(j\omega) \right] \) AND \( \sigma_{\max} \left[ W_m(j\omega) \right] \)
ROBUST STABILITY TEST: USED CLOSED-LOOP TFM

\[ \sigma_{\text{max}} \left( \frac{1}{w(j\omega)} \right) \]

\[ m \]

\[ \log \omega \]

\[ 0 \text{ db} \]
\[
\max_w \left[ \sigma_{\text{max}} \{ (I+\mathbf{GK})^{-1} \mathbf{GK} \cdot \mathbf{W}_n(s) \} \right] < 1
\]

This guarantees robust stability.

i.e. if, \[ \left\| (I+\mathbf{GK})^{-1} \mathbf{GK} \mathbf{W}_n \right\|_{\infty} < 1 \]

So far we have been able to come-up with two TEM to shape:

(1) The sensitivity (one equiv. the \textit{loop}) in order to guarantee nominal performance.

The shaping function (or filter) \( W_p(s) \) was primarily determined by the performance spec. at low freqs. i.e. low frequency barrier and the performance rejection of bandwidth constraints at high frequency (high frequency barrier).

\[ 0 \leq \sigma(s(j\omega)) \leq \frac{1}{W_p(s)} \]
(2) The complementary sensitivity (or closed-loop) in order to guarantee robust stability to modeling errors described by $S(s)$. The shaping function (or filter) $W_M(s)$ is primarily determined by the modeling errors of the desired GM (PM's).

Note that since $S(s) + T(s) = I$ we may end-up asking the controller to do things it cannot. (i.e. pick $W_P(s)$ of $W_M(s)$ carefully).
For robust performance we define a new spec. now, that is the tracking error of the controller, we want

\[ \sigma_{\text{max}} \left| \frac{e(s)}{r(s)} \right| \leq \frac{1}{W_p(s)} \]

where \( e(s) = r(s) - y(s) \) and \( W_p(s) \) is the performance filter we talked about before.

Figure of the block diagram.

If the nominal loop is stable, the robust performance is possible if:

\[
\left| \frac{GK \cdot W_m}{1+GK} \right| + \left| \frac{W_p}{1+GK} \right| < 1
\]

\( \forall e, r \) for SISO case.

In MIMO, this becomes a bit more complicated:

\[
\left[\begin{array}{ccc}
\varepsilon (I+GK)^{-1} W_p & - (I+GK)^{-1} W_m \\
GK(I+GK)^{-1} W_p & -GK(I+GK)^{-1} W_m \\
\end{array}\right] < 1, \quad \forall e, r
\]
ROBUST PERFORMANCE

CONSIDER ROBUST PERFORMANCE UNDER MULTIPLICATIVE UNSTRUCTURED UNCERTAINTY

\[ \text{Spec: } |e(j\omega)| \leq \frac{1}{W_p(j\omega)} \]

IF THE NOMINAL LOOP IS STABLE THEN ROBUST PERFORMANCE IS POSSIBLE IF

\[
\frac{GK W_m(j\omega)}{1 + GK(j\omega)} + \frac{W_p(j\omega)}{1 + GK(j\omega)} < 1
\]

WHERE \[ \sigma_{\max} [\Delta(j\omega)] \leq W_m(j\omega) \]
Implications:

Before robust performance we asked for:

\[ \| T(s) \cdot W_m \|_{\infty} < 1 \]

\[ \| S(s) \cdot W_p \|_{\infty} < 1 \]

Now we are asking (in \( s(\infty) \))

\[ \| T(s) \cdot W_m \| + \| S(s) \cdot W_p \| < 1 \]

(More conservative design but robust in performance.)
Open-loop Shaping

Closed-loop Shaping

Nom. Performance

\[ \sigma(\sigma(j\omega)) \]

\[ \sigma(\sigma(j\omega)) \]

\[ \max \{ \sigma_{\text{max}} \{ S(j\omega)W_p(j\omega) \} \} < 1 \]

\[ \max \{ \sigma_{\text{max}} \{ C(j\omega)W_m(j\omega) \} \} < 1 \]

\[ \| C(j\omega)W_m(j\omega) \|_{\infty} < 1 \]

Robust Stability

\[ \sigma[C(j\omega)] \]

\[ \sigma[C(j\omega)] \]

\[ \| C(j\omega)W_m(j\omega) \|_{\infty} < 1 \]

Robust Performance

\[ \sigma(S(j\omega)) \]

\[ \sigma(S(j\omega)) \]

\[ \max \{ \sigma_{\text{max}} \{ \sigma(S(j\omega)W_p(j\omega) \} \} \]

\[ \max \{ \sigma_{\text{max}} \{ 1 - |\sigma(S(j\omega)C(j\omega))| \} \} \]

\[ \| S(j\omega)W_p(j\omega) + W_m(j\omega)C(j\omega) \| < 1 \]

Nom. Stability Implicit!
• EXAMPLE

\[ G(s) = \frac{1}{1 + 2\zeta(s/\omega_0) + (s/\omega_0)^2} \cdot \left( \frac{-s + 5}{s + 5} \right) \]

where \( \zeta = 0.1, \omega_0 = 1 \)

Let the controller \( K(s) \) be given by

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -0.2 & 1 & -1 \\
0 & 0 & -5 & 10 \\
1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}
\]

\[ K(s) = \frac{-5(s + 0.2)(s + 0.1)}{(s + 5)s} = \begin{bmatrix}
-5 & 1 & 1 \\
0 & 0 & 0.1 \\
24 & -5 & -5
\end{bmatrix} \]

• NOMINAL STABILITY

CLOSED-LOOP POLES AT -7.82, -0.5 + j 1.65, -1.36, and -0.016 rad/sec. ALL IN THE LHP, SO NOMINAL STABILITY ACHIEVED.
• NOMINAL PERFORMANCE

Nominal Performance Test

\[ \text{G} \times \text{inv}(I - \text{KG}) \]

Frequency (rad/s)

• ROBUST STABILITY

Robust Stability Test

\[ \text{W}_1 \times \text{K} \times \text{inv}(I - \text{GK}) \]

Frequency (rad/s)
• ROBUST PERFORMANCE

\[ \rho_0 \leq \mu \leq \sigma_{\max} \]

Robust Performance Test

\[ y \]

\[ K \]

\[ G \]

\[ W_1 \]

\[ \Delta_1 \]

\[ W_2 \]

\[ \Delta_2 \]
Topics to be covered:

- The Model-based Compensators & the LQG algorithms
- The MBC & the Loop Transfer Recovery (LTR)
- The accurate measurement Kalman Filter (KF)
- The KF frequency-domain equality
- Shaping the SV of the KF loop transfer matrix

Ideally would like to cover:

- State-Variable Feedback - Properties, etc., LQR, etc.
- Time Varying LQ, Disturbance Rejection, Derivatives of LQG, etc.
- MBC & the LQG Algorithm / H∞ Algorithm
- The Kalman Filter & its Derivatives (Accurate measurement problem)
- The KF frequency-domain equality
- LQ/Servo
- MBC & LTR
- Shaping of the SV & the KF LTR - The tricks
PROPERTIES OF STATE VARIABLE

FEEDBACK

Assume,
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Ce^{x(t)} \]
\[ \in \mathbb{R}^n, \, u \in \mathbb{R}^m, \, y \in \mathbb{R}^p. \] The system \( 1 \) represents the open-loop dynamics. In the frequency domain \( \omega \) is described as:

\[ X(s) = (sI-A)^{-1}B \, u(s) \]
\[ Y(s) = C(sI-A)^{-1}B \, u(s) \]

It is useful to think of the physical meaning of state variables as those variables whose knowledge at each instant of time characterizes the energy stored in the system; physical state variables are inductor currents, e.g, voltages, positions, accelerations, speeds, pressures, etc., since they define energy. If such physical state variables are directly available for measurement, one can adjust the controls (which absorb or introduce energy to the system) as a function of the state variables so as to make the system outputs have desirable time-response characteristics.
If we use linear state feedback, given by
\[ u(t) = -G x(t) + u_c(t) \quad (3) \]

where \( G \) is a constant \( m \times n \) gain matrix. In general, \( u_c(t) \) is designed for good command following & disturbance rejection.

The closed-loop system is described as follows:
\[
\begin{align*}
\dot{x}(t) &= \left[ A - BG \right] x(t) + B u_c(t) \\
y(t) &= C x(t)
\end{align*}
\quad (4)
\]

In the frequency domain:
\[
\begin{align*}
x(s) &= \left( sI - A + BG \right)^{-1} B u_c(s) \\
y(s) &= C \left( sI - A + BG \right)^{-1} B u_c(s)
\end{align*}
\quad (5)
\]

Clearly the eigenvalues \( \lambda_i(A) \), \( i = 1, \ldots, n \) are the open-loop poles & the eigenvalues \( \lambda_i(A - BG) \), \( i = 1, \ldots, n \) are the closed-loop poles.
The following facts are known from linear theory:

**Theorem 1:** If \([A, B] \) is controllable, then there exists at least one gain matrix \( C \) so that the closed-loop poles \( \lambda_i(A - BC) \) can be arbitrarily placed in the \( s \)-plane; of course, complex closed-loop poles must be in complex conjugate pairs.

The implication of this theorem is that by appropriately selecting \( C \) we can always get a stable closed-loop system. This is true even if all of the poles of \( A \) are on the RHP.

Another property of state feedback is that it does not add or change the location of the open-loop zeros.

**Theorem 2:** The zeros of the open-loop transfer matrix \( G_0(s) \) defined by:

\[
G_0(s) = C (sI - A)^{-1} B
\]

are identical to the zeros of the closed-loop transfer matrix, defined by:

\[
G_c(s) = C (sI - A - BC)^{-1} B
\]
Interpretation of the State Variable Feedback Method as Compensation

We can visualize the effects of state feedback as defining the LTI for a standard MIMO feedback configuration.

\[ M_c(s) \rightarrow M_c(s) \rightarrow (S I - A)^{-1}B \rightarrow G \rightarrow X(s) \rightarrow G \rightarrow M_c(s) \]

If we define the LTI, \( G_{\text{loop}}(s) \) by:

\[ G_{\text{loop}}(s) = G (S I - A)^{-1} B \]

Then,

\[ M_c(s) \rightarrow M(s) \rightarrow G_{\text{loop}}(s) \rightarrow M(s) \rightarrow M_c(s) \]

Now, we can use all the machinery we built for robustness and stability using the SIVs.

It is important to notice that use of state variables feedback does not provide dynamic compensation. The importance of this remark will become obvious when we try to shape the loop dynamics. Augmenting the open-loop dynamics \((S I - A)^{-1} B\)
with additional dynamics allows for superior command input tracking and disturbance rejection.

The main value of state variable feedback is that it stabilizes a control system. However, the remaining measures of goodness are still not answered.

Even though we know that we can stabilize a system, we still don't know how to pick good G's. That's why we try the LQ design method.
THE LINEAR QUADRATIC REGULATOR (LQR) PROBLEM

In this note we summarize the solution and properties of the Linear Quadratic Regulator problem.

The style followed here is to expose the properties of the LQR problem with respect to

1. Stability of the Closed Loop Control System
2. Solution to a Dynamic Optimal Control Problem
3. Inherent Robustness Properties in terms of Singular Values of the resultant Return Difference Matrix through LQR compensation.

1. The Open Loop System

In the time-domain the open-loop system is described in state variable form as follows:

\[ \dot{x}(t) = A x(t) + B u(t) \]

\[ y(t) = C x(t) \]

with \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \).

2. Assumptions

We make the following assumptions

Assumption 1. The entire state vector \( x(t) \) is available for feedback.
Assumption 2. The system (1) is stabilizable. Recall that a system is stabilizable if all of its unstable modes are controllable. We also remark that if \([A, B]\) is controllable, then the system (1) is stabilizable; thus controllability is a stronger assumption than stabilizability.

Assumption 3. The system (1) is detectable. Recall that a system is detectable if all of its unstable modes are observable. We also remark that if \([A, C]\) is observable, then the system (1) is detectable; thus observability is a stronger assumption than detectability.

3. The Algebraic Riccati Equation (ARE)

Let \(R\) be an arbitrary symmetric positive definite \(m \times m\) matrix, i.e.

\[
R = R' > 0 ; \quad R^{-1} \text{ exists}
\]  

Then consider the following matrix ARE

\[
O = -KA - A'K - C'C + KBR^{-1}B'K
\]  

Equation (3) may be frightening at first but it is nothing more than a shorthand for a bunch of complex algebraic quadratic equations in the elements \(K_{ij}\) of the \(n \times m\) matrix \(K\). Think of \(A, B, C\) and \(R\) as being the coefficients and that the matrix \(K\) is being the "unknown".

The ARE has several important properties, that we outline below.

Property 1: The solution matrices are symmetric, i.e.

\[
K = K'
\]  

This can be readily verified by taking the transpose of both sides of Eq (3).

Property 2. In general, there is a very large number of matrices \(K\) that are solutions to the ARE. However, if \([A, B]\) is controllable and \([A, C]\) is observable, then there exists one and only one solution matrix \(K\) to the ARE that is positive definite, i.e.

\[
K = K' > 0
\]
If we relax the controllability assumption to that of stabilizability, and the observability assumption to that of detectability, then there exist one and only one positive semidefinite solution matrix to the ARE, i.e.

\[ K = K^t > 0 \]  

(6)

Remark: Given \( A, B, C, \) and \( R \) one needs a computer to find the positive (semi) definite solution matrix \( K \) to the ARE which is important for designing stable control systems. Modern software* find the desired solution matrix directly; they do not find all solution matrices and check for condition (5) or (6)!

Henceforth, when we refer to the solution matrix \( K \) to the ARE we shall imply the unique positive (semi) definite one.

The LQ Control Gain Matrix

Once we are given \( A, B, C \) and \( R \) then we can calculate the solution matrix \( K \) of the ARE. On the basis of this calculation we define the LQ control gain matrix \( G \) as follows:

\[ G = R^{-1}B'K \]  

(7)

Clearly \( G \) is a constant \( m \times n \) matrix.

The LQR Control Law

The LQR control law utilizes full state variable feedback for the system (1) using the control gain \( G \) calculated according to Eq. (7). Thus, the control law is

\[ u(t) = -Gx(t) \]  

(8)

The LQR Closed Loop System

Substituting Eq (8) into Eq (1) we readily deduce that the closed-loop system evolves according to the state equations

\[
\begin{align*}
\dot{x}(t) &= [A-B \ G]x(t) \\
y(t) &= C \ x(t)
\end{align*}
\]  

(9)

The LQR design has several very important properties which we shall summarize in the sequel. It is important to stress that the properties of LQR designs hinge upon

1. the fact that full-state feedback is used
2. the specific way that the control gain matrix \( G \) is computed via the solution matrix \( K \) of the ARE.

Furthermore, it is important to stress that these properties hold

1. for any order system, where the order of the system, \( n \), is the dimensionality of the state vector
2. for any number of controls, \( m \), and outputs, \( p \),
3. for any matrices \( A, B, C \) (modulo the stabilizability and detectability assumptions) and \( R = R' > 0 \). Thus the open-loop system may have several unstable open-loop poles and several nonminimum phase zeros.

Property 1

**Guaranteed Stability of the LQR Closed Loop System**

The poles of the closed-loop system (9) are strictly in the left half of the s-plane and hence the closed-loop system is asymptotically stable, i.e.

\[ \text{Re } \lambda_i [A-B \ G] < 0 \]  

(10)
Property 2: Optimality

The LQR control law (8) generates the minimum possible value of the quadratic cost functional $J$ given by

$$J = \int_0^\infty [y'(t)\dot{y}(t) + u'(t)R\ u(t)]\,dt$$  \hspace{1cm} (11)

subject to the dynamic constraints imposed by the open-loop dynamics (1).

Property 3: Excellent Robustness Properties

We have seen that any state variable feedback compensation scheme generates a specific loop transfer matrix. Since we have computed the LQR control gain matrix in a specific way, we are going to get a specific loop transfer matrix for any given $A$, $B$, $C$, and $R$. We can represent the closed-loop system given by Eq. (9) as shown in Fig. 1,

\[ G(s) = \frac{C(sI-A)^{-1}B}{s} \]

FIG. 1: The LQR loop transfer matrix

where the loop transfer matrix, denoted by $G_{LQ}(s)$, induced by the LQR design scheme is given by

$$G_{LQ}(s) = G(sI-A)^{-1}B$$  \hspace{1cm} (12)

Recall that the multivariable robustness properties of any design depend on the size of

$$\sigma_{\min}[I+G(j\omega)] \text{ and } \sigma_{\min}[I+G(j\omega)]$$  \hspace{1cm} (13)

Now we make the following additional assumption:
Assumption 4. The matrix $R = R' > 0$ is also diagonal.

Under the above assumption we have the following guaranteed LQR robustness properties

\[
\begin{align*}
\sigma_{\min} (I+G_{LQ}(j\omega)) & \geq 1 \quad \text{for all } \omega \\
\sigma_{\min} (I+G_{LQ}^{-1}(j\omega)) & \geq \frac{1}{2} \quad \text{for all } \omega
\end{align*}
\]

From the inequalities (14) and (15) we can deduce the following guaranteed multivariable gain and phase margin properties of any LQR design

\(1\) Upward Gam Margin is infinite
\(2\) Downward Gam Margin is at least $\frac{1}{2}$ (or -6db)
\(3\) Phase Margin is at least $+60\%$.

We remind the reader that the gain margins $[\frac{1}{2}, \infty]$ can occur independently and simultaneously on all $m$ control channels. Similarly, the phase margins $[+60^\circ, -60^\circ]$ can occur simultaneously and independently on all control channels.

These inherent robustness properties of LQR design are quite good for many applications. To appreciate what they mean let us consider a SISO system

\[
\begin{align*}
\dot{x}(t) &= A x(t) + b u(t) \\
y(t) &= c'x(t)
\end{align*}
\]

and let $R$ be a positive scalar. Then the LQ control gain is simply an $n$-vector, $g$, given by

\[
g = \frac{1}{R} b' K
\]

the control law is

\[
u(t) = -g'x(t)
\]
Fig 2. Every SISO LQR design leads to a loop transfer function that avoids the unit circle about the critical point in the Nyquist diagram.
and the resulting LQ loop transfer function

\[ g_{LQ}(s) = g'(sI-A)^{-1}d \]  

(19)

We can now visualize the robustness properties in the ordinary Nyquist diagram by plotting \( g_{LQ}(j\omega) \) for all \( \omega \). The visualization of the inherent robustness properties is shown in Fig. 2; the Nyquist locus \( g_{LQ}(j\omega) \) is guaranteed not to get inside the unit circle centered at the (-1,0) point. This is why the LQR obtains the good gain and phase margins.

\[ \text{Property 4: LQR Rolloff Characteristics} \]

For SISO LQR designs Fig. 2 shows that as \( \omega \to \infty \) the phase of \( g_{LQ}(j\omega) \) cannot be less than -90°. This implies that the high frequency behavior of \( g_{LQ}(j\omega) \) is of the form

\[ \lim_{\omega \to \infty} |g_{LQ}(j\omega)| \sim \frac{1}{\omega} \]  

(20)

so that SISO LQR designs exhibit "an one-pole rolloff" behavior. This property is also true for MIMO designs in the sense that

\[ \lim_{\omega \to \infty} |G_{LQ}(j\omega)| \sim \frac{1}{\omega} \]  

(21)

Strictly speaking, this violates the Bode-Horowitz condition that all physical systems must exhibit at least a 2-pole rolloff behavior. The reason that we get an one-pole rolloff characteristic in LQR designs is due to our assumption that we have measured all state variables and we can feed them back. If we have unmodeled dynamics (i.e. states that have not been included in the state space description (1)) then all the properties that we have discussed above are not necessarily true nor guaranteed.
If the unmodeled dynamics are high-frequency ones, and if the LQR design leads to a stable closed-loop system, then we would actually get at least a 2-pole rolloff. In SISO designs, in reference to Fig. 2, this means that the actual $g_{LQ}(j\omega)$ will penetrate the unit disk about the (-1,0) point at high frequencies; this in turn will imply that real LQR designs will not have the infinite upward gain margin property.

The above remarks are not meant to degrade the power of LQ-based designs. At this point they are intended to serve as a warning to the reader that mathematical results, derived on the basis of certain assumptions, should be carefully scrutinized from a pragmatic control system point of view. A good control system designer should be intimately familiar with both the strengths and weaknesses of a computer-aided design procedure. Fortunately, as we shall see, we can "tune" LQ-based designs so that their shortcomings in practical designs can be tolerated. After all, demanding an upward gain margin does not make much engineering sense. Control system designers should be very happy with upward gain margins of the order tens of db's; otherwise, they must admit that they are truly sloppy modellers!!

In closing, we remind the reader that LQR-based designs are special cases of general full state variable feedback designs; consequently they have the following general properties.

(a) LQR designs do not introduce extra dynamics in the loop transfer matrix.

(b) LQR designs do not change the location of the open-loop transmission zeros nor do they create any new ones.

* This can be done by controlling the gain crossover frequency in LQR designs; we shall elaborate at length upon this point in subsequent notes.
6.232 MULTIVARIABLE CONTROL SYSTEMS
SPRING 1985

VARIANTS OF THE LINEAR QUADRATIC
REGULATOR (LQR) PROBLEM

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March 18 1985

REF. NO. 850318/6232

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VARIANTS OF THE LINEAR QUADRATIC REGULATOR (LQR) PROBLEM

0. Summary

1. The objective of the lecture note is to summarize some useful variants of the LQR problem which often turn out to be useful.

1. The first problem deals with the optimal control of an LTI system with respect to a quadratic performance criterion which contains cross-penalty terms in the state and control vectors.

2. The second problem formulation deals with adding an exponential time weighting term in the standard quadratic criterion. As we shall see, one can select a design parameter so that the LQR closed-loop system has a prescribed degree of stability.
The third problem is a stochastic version of the LQR problem. In this formulation we allow the state dynamics to include the effects of white process noise. As we shall see, the deterministic structure of the LQR problem remains optimal in the stochastic case provided that optimality is suitably defined.

1. LQR Problem with Cross Penalty

1.1 Problem Formulation

Consider an LTI system with state equation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

and the following quadratic cost functional

\[ J = \int_0^\infty \left[ x'(t)^T Q x(t) + 2 x'(t)^T R u(t) + u'(t)^T S u(t) \right] dt \]

The standard LQR problem was solved under the assumption \( S = 0 \).

In this problem formulation we make the
following assumptions:

\[ Q = Q' > 0 \]  \hspace{1cm} (1.2) \\
\[ R = R' > 0 \]  \hspace{1cm} (1.3) \\
\[ [A, B] \text{ is stabilizable} \]  \hspace{1cm} (1.4) \\
\[ [A, Q^{1/2}] \text{ is detectable} \]  \hspace{1cm} (1.5) \\

\[
\begin{bmatrix}
  Q & S' \\
  S' & R \\
\end{bmatrix} \geq 0
\]  \hspace{1cm} (1.6)

We remark that assumptions (1.2) to (1.5) are identical to those in the standard LQR problem.

The additional assumption (1.6) simply guarantees that the integrand of the cost function (1.1) is non-negative.

1.2 Motivation

Let us examine two problems that give rise to the problem formulation above.

Problem A

Consider an LTI system with the property that
the control \( u(t) \) feeds through to the output \( y(t) \), i.e.

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &=Cx(t) + Du(t) \quad ; \quad D \neq 0
\end{align*}
\]

(1.7)

Consider the following cost functional

\[
J = \int_0^\infty \left[ y'(t)y(t) + u'(t)R u(t) \right] dt
\]

(1.8)

Substituting the output equation into the cost functional (1.8) we obtain

\[
J = \int_0^\infty \left[ x'(t)C'C x(t) + 2x'(t)C'D u(t) \\
+ u'(t)\left[D'D + R\right] u(t) \right] dt
\]

(1.9)

which clearly has the structure of the cost functional (1.1)

**Problem B**

\( q \) Sometimes we may wish to penalize the derivatives \( \dot{x}_i(t) \) of some (or all) state variables in the quadratic cost functional.
This will make the regulator more sluggish and the trick is used to create a more "damped" closed-loop response. So let us suppose that the state dynamics are given by

\[ \dot{x}(t) = A x(t) + B u(t) \]  \hspace{1cm} (1.10)

and the cost functional has the form

\[ J = \int_0^\infty \left[ \dot{x}'(t) \dot{x}(t) + u'(t) R u(t) \right] dt \]  \hspace{1cm} (1.11)

After substituting eq. (1.10) into the cost functional (1.1), we obtain

\[ J = \int_0^\infty \left\{ \dot{x}'(t) A' A x(t) + 2 \dot{x}'(t) A' B u(t) + u'(t) \left[ B' B + R \right] u(t) \right\} dt \]  \hspace{1cm} (1.12)

Once more we see that the structure of the cost functional (1.12) exhibits the cross terms of the cost functional (1.1).
1.3 Problem Solution

We now summarize the solution of the optimal control problem posed in Section 1.1. We remark that the solution can be derived by variational techniques very similar to those used in the standard LQR problem. The proof is left as an exercise for the reader.

The optimal control can be implemented by full-state feedback and is given by

$$U(t) = -G_x(t)$$  

(1.13)

The control gain matrix $G$ is given by

$$G = R^{-1}[S' + B'K]$$  

(1.14)

The matrix $K = K' > 0$ is computed from the solution of the following algebraic Riccati equation

$$0 = -KA - A'K - Q + [KB + S']R^{-1}[B'K + S']$$  

(1.15)
1.4 Discussion

The resultant closed-loop regulator is asymptotically stable, i.e.

\[ \text{Re } \gamma \text{ } [A - BG] < 0 \]  \hspace{1cm} (1.16)

However, the robustness results derived for the standard LQ regulator are not necessarily true.

2. LQR Problem With Exponential Penalties

2.1 Problem Formulation

We now consider the following variant of the standard LQR problem. The state dynamics are given by

\[ \dot{x}(t) = Ax(t) + Bu(t) \] \hspace{1cm} (2.1)

and the cost functional is given by

\[ J = \int_0^\infty e^{2\alpha t} \left\{ x'(t) Q x(t) + u'(t) R u(t) \right\} dt \] \hspace{1cm} (2.2)

with \( \alpha > 0 \) \hspace{1cm} (2.3)
The remaining assumptions are those for the standard LQR problem (see eqs. (1.2) to (1.5)).

Some thought on the part of the reader should suggest what is the impact of the exponential weighting in the cost functional (2.2). In order for the cost to be minimum, $J$ has to be finite. Since $\alpha > 0$, this implies that the quadratic integrand must decrease faster than $\exp \{-2\alpha t\}$, i.e.

$$x'(t)^TQx(t) + u'(t)^TRu(t) \leq M e^{-2\alpha t}$$

(2.4)

where $M$ is some positive constant. Hence, we should suspect that the closed-loop poles cannot be in arbitrary locations in the left half of the $s$-plane. The precise properties of the closed-loop poles will be discussed in the sequel.
2.2 Problem Solution

We now summarize the solution to the optimal control problem defined in Section 2.1.

The optimal control can be realized by LTI state variable feedback and is of the form

\[ u(t) = -Gx(t) \]  \hspace{1cm} (2.5)

The control gain matrix \( G \) is given by

\[ G = R^{-1}B'K \]  \hspace{1cm} (2.6)

The matrix \( K = K' > 0 \) is the unique solution of the following algebraic Riccati equation

\[ 0 = -K(A+\kappa I) - (A+\kappa I)'K - Q + KBR^{-1}B'K \]  \hspace{1cm} (2.7)

Note that the ARE can be easily evaluated with standard computer-aided design software.

2.3 Solution Properties

The major impact of the inclusion of the weighting \( \exp\{2\kappa t\} \) in the quadratic cost function (2.2) relates to the location of the
poles of the closed-loop regulator

\[ \dot{x}(t) = [A - BG]x(t) \]  \hspace{1cm} (2.8)

\# No matter what are the numerical values of 
\[ A, B, Q, \text{ and } R \] (modulo our standing LQG assumptions) it is true that

\[ \text{Re } \lambda_i [A - BG] < -\infty \]  \hspace{1cm} (2.9)

as illustrated in Fig. 2.1

![Region of closed-loop poles](image)

**Fig 2.1** Region of closed-loop poles in the S-plane

Thus, both the closed-loop state \( x(t) \), and closed-loop control, \( u(t) \), decay faster.
It is easy to demonstrate that all the other inherent properties of LQR design (with $R = \text{diagonal matrix}$), such as the multivariable gain and phase margin properties and the singular value inequality properties remain valid in this case as well.

2.4 Elements of Proof

The solution of the optimal control problem can, of course, be derived with variational arguments similar to those used to derive the standard LQR problem.

However, there is a more direct way that can be used to derive the solution summarized in Section 2.2. The idea is to make a change of variables so that the problem definition of Section 2.1 reduces to that of a standard LQR problem.
Consider the following variable definition
\[ \tilde{x}(t) \triangleq e^{at} x(t) \] (2.10)
\[ \tilde{v}(t) \triangleq e^{at} u(t) \] (2.11)

Clearly
\[ x(t) = e^{-at} \tilde{x}(t) \quad ; \quad u(t) = e^{-at} \tilde{v}(t) \] (2.12)

and so the cost functional (2.2) reduces to
\[ J = \int_0^\infty \left[ \tilde{x}'(t)Q \tilde{x}(t) + \tilde{v}'(t)R \tilde{v}(t) \right] dt \] (2.13)

which of course has the structure of the quadratic cost in the standard LQR problem.

Now let us compute the dynamics of \( \tilde{x}(t) \).

From eq. (2.10) we have
\[ \dot{\tilde{x}}(t) = \alpha e^{at} \tilde{x}(t) + e^{at} \dot{x}(t) \]
\[ = \alpha \tilde{x}(t) + e^{at} \left[ A \tilde{x}(t) + B u(t) \right] \]
\[ = \alpha \tilde{x}(t) + A \tilde{x}(t) + B \tilde{v}(t) \]
\[ = \left[ A + \alpha I \right] \tilde{x}(t) + B \tilde{v}(t) \] (2.14)

Thus we want to solve the standard LQR problem for the system
\[ \dot{\tilde{x}}(t) = \left( A + \alpha I \right) \tilde{x}(t) + B \tilde{v}(t) \] (2.15)
with respect to the cost functional (2.13). The results of Section 2.2 follow directly once we transform the solution of the standard LQR problem to the original variables via eq. (2.12).

3. The Stochastic LQR Problem

3.1 Problem Formulation

In the stochastic LQR problem we allow the inclusion of white process noise in the state dynamics. We still assume that all state variables can be measured exactly, and can be used for feedback as necessary.

We assume that the state vector $\mathbf{x}(t)$ satisfies the stochastic differential equation

$$\dot{\mathbf{x}}(t) = A \mathbf{x}(t) + B \mathbf{u}(t) + \mathbf{w}(t)$$

(3.1)

where $\mathbf{w}(t)$ is a zero-mean white noise process.

We seek the control that minimizes the following quadratic cost function.
\[
J = \lim_{T \to \infty} \frac{1}{T} E \int_0^T \left[ x'(t) @ x(t) + u'(t) R u(t) \right] dt \tag{3.2}
\]

The integrand of the cost functional is identical to that of the deterministic LQR problem and we make the same assumption (see eqs (1.2) to (1.5)). The difference is that the integrand is now a stochastic process so that we must minimize its expected value \( E \{ \cdot \} \). The normalization by the integration time \( T \) is necessary to ensure that the cost functional is finite.

3.2 Problem Solution

Surprisingly the optimal control for the stochastic LQR problem is identical to that of the corresponding deterministic LQR problem. This result hinges upon the assumption that \( S(t) \) is white. The proof is quite complicated if one insists on rigor; it requires advanced mathematics related to stochastic.
and is beyond the scope of this book. For the sake of completeness we state the solution to the stochastic LQR problem below.

The optimal control is

\[
\mathbf{u}(t) = -\mathbf{G} \mathbf{x}(t)
\]  \hspace{1cm} (3.3)

where

\[
\mathbf{G} = \mathbf{R}^{-1} \mathbf{B}' \mathbf{K}
\]  \hspace{1cm} (3.4)

and \( \mathbf{K} = \mathbf{K}' > \mathbf{0} \) is the solution matrix of the algebraic Riccati equation

\[
0 = -\mathbf{K} \mathbf{A} - \mathbf{A}' \mathbf{K} - \mathbf{Q} + \mathbf{K} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}' \mathbf{K}
\]  \hspace{1cm} (3.5)

3.3 Properties

All properties of the deterministic LQR problem are also true for the stochastic LQR problem.
The purpose of this note is to summarize the optimal solutions to the so-called LQ tracking problem and to the LQ disturbance rejection problem. These problems are important because they expose the relationship between the LQ regulator problem to problems in which deterministic reference input tracking (command following) and deterministic disturbance rejection are appropriately formulated as optimal control problems.

1. The LQ Reference Input Tracking Problem

1.1 Problem Formulation

We start with the state-space description of the open-loop dynamics

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t); \quad x(t_0) = x_0
\]

(1)

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m. \)

*No proofs will be presented. An appropriate reference is M. Athans and P.L. Falb, Optimal Control, McGraw-Hill Co., 1966, pp. 793-814. The derivations there rely upon the maximum principle; however, they can be derived using the calculus of variations method, and you will be asked to prove the results in this manner.*
Let us suppose that $\mathbf{r}(t) \in \mathbb{R}^p$ is a known deterministic vector-valued time function, called the reference input. We stress that we make the assumption that $\mathbf{r}(t)$ is known for all $t$, $t_0 \leq t \leq t_f$, where $t_f$ is the final time.

Let the output vector of the open-loop system be denoted by $y(t)$. In analogy with ordinary servomechanism problems we want to have the output $y(t)$ follow the reference input $\mathbf{r}(t)$ with small error for all $t$, $t_0 \leq t \leq t_f$. Thus, $\mathbf{r}(t)$ and $y(t)$ must have the same dimension. Thus we write

$$\mathbf{y}(t) = \mathbf{c}(t)\mathbf{x}(t)$$

with $\mathbf{y}(t) \in \mathbb{R}^p$, and we can define the error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t) = \mathbf{r}(t) - \mathbf{c}(t)\mathbf{x}(t)$$

In analogy with the LQ regulator problem we can translate our reference input tracking problem into the minimization of the following quadratic index

$$J = \frac{1}{2} \mathbf{e}'(t_f) \mathbf{F} \mathbf{e}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ \mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) \right] dt$$

with

$$\mathbf{F} = \mathbf{F}' > 0; \quad \mathbf{Q}(t) = \mathbf{Q}'(t) > 0; \quad \mathbf{R}(t) = \mathbf{R}'(t) > 0$$

The dynamic optimal control problem then consists of finding the optimal control $\mathbf{u}(t)$, $t_0 \leq t \leq t_f$, that minimizes the cost functional (4) subject to the constraints (1), (2), (3), and (5).
1.2 The Structure of the Optimal Control

The optimal control exists, is unique, involves full-state variable feedback, and it has the following structure

\[ u(t) = -G(t)x(t) + u_c(t) \]  \hspace{1cm} (6)

where \( G(t) \) is an \( mxn \) matrix of control feedback gains.

The point that we want to stress at this point is that at each instant of time, \( t \), the optimal control involves

(a) instantaneous state variable feedback
   - the \( G(t)x(t) \) term in eq. (6), and

(b) a time-correction term \( u_c(t) \).

As we shall see below, the control gain matrix \( G(t) \) is independent of the nature or dimension of the reference input \( \bar{r}(t) \). The detailed time-behavior of \( \bar{r}(t) \) is captured in the \( u_c(t) \) term of eq. (6).

1.3 Calculation of the Control Gain Matrix \( G(t) \)

The control gain matrix \( G(t) \) in eq. (6) is given by

\[ G(t) = R^{-1}(t)B'(t)K(t) \]  \hspace{1cm} (7)

where the \( nxn \) matrix \( K(t) \) is the unique, symmetric, at least positive semidefinite solution matrix of the matrix Riccati differential equation

\[ \dot{K}(t) = -K(t)A(t) - A'(t)K(t) - C'(t)Q(t)C(t) \]
\[ + K(t)B(t)R^{-1}(t)B'(t)K(t) \]  \hspace{1cm} (8)

subject to the boundary condition at the terminal time \( t_f \)

\[ K(t_f) = C'(t_f)F^{-1}C(t_f) \]  \hspace{1cm} (9)
Note that eqs. (8) and (9) do not depend upon the reference input \( \mathbf{r}(t) \); hence, the matrix \( \mathbf{K}(t) \) and the gain matrix \( \mathbf{G}(t) \) do not depend upon the reference input \( \mathbf{r}(t) \). This establishes the remark made at the end of Section 1.2.

1.4 Calculation of the Control Correction Term \( \mathbf{u}_c(t) \)

The control correction term \( \mathbf{u}_c(t) \), \( \mathbf{u}_c(t) \in \mathbb{R}^m \), is given by the equation

\[
\mathbf{u}_c(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t)
\]

where \( \mathbf{g}(t) \in \mathbb{R}^n \) is the solution vector of the linear vector differential equation

\[
\dot{\mathbf{g}}(t) = -[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]' \mathbf{g}(t) - \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{g}(t)
\]

subject to the boundary condition at the terminal time \( t_f \)

\[
\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}\mathbf{r}(t_f)
\]

Let us highlight the implications of the above result. The present value of the vector \( \mathbf{g}(t) \) is determined by the solution of (11); however, in view of the terminal time boundary condition (12), the present value of \( \mathbf{g}(t) \), and hence of \( \mathbf{u}_c(t) \) and hence of \( \mathbf{u}(t) \), depends upon the future values of the reference input \( \mathbf{r}(\tau) \), \( t \leq \tau \leq t_f \).

The above may sound strange at first sight but it makes perfectly good sense. We "told" the mathematics that we knew the entire time-trajectory of the reference input \( \mathbf{r}(t) \), for all \( t_0 \leq t \leq t_f \). The control correction term \( \mathbf{u}_c(t) \) adjusts the instantaneous feedback term \(-\mathbf{G}(t)\mathbf{x}(t)\) to take into account the future time evolution of the specific reference input that we want to track. Remember: the optimal control now represents the "best decisions" now to make the future errors small. Hence, it has an anticipatory term. In the optimal control equation

\[
\mathbf{u}(t) = -\mathbf{G}(t)\mathbf{x}(t) + \mathbf{u}_c(t)
\]
(1) the current state $x(t)$ summarizes the past.

(2) the current control gain $G(t)$, via eqs. (7) to (9) takes into account the future system dynamics via $A(\tau), B(\tau), C(\tau)$ for $\tau \leq t \leq t_f$, and the future way of penalizing errors and controls via the weighting matrices $F, Q(\tau), R(\tau)$ for $\tau \leq t \leq t_f$ independent of the reference input.

(3) the current control correction term $u_{c}(t)$, via eq. (10) takes into account both the future system dynamics and weighting matrices and the future time dependence of the reference input $\bar{r}(\tau), t \leq \tau \leq t_f$.

At this point the practically-oriented reader may state: This is all and good, but in most practical problems I do not know the future of my reference input because it may be generated by an exogenous source*; so what do I do? The answer is that one cannot formulate a deterministic optimal control problem; the statement "I do not know" must be suitably translated to a stochastic optimization problem. We shall see how this can be done in subsequent lecture notes.

1.5 The Output Regulator Problem

If the reference input $\bar{r}(t)$ is identically zero for all $t_0 \leq t \leq t_f$, i.e.

$$\bar{r}(t) = 0; \quad t_0 \leq t \leq t_f$$

(14)

then the error $e(t) = -\bar{r}(t)$ simply represents the deviation of the output $y(t)$ from the reference vector $\bar{r}$. Substituting eq. (14) into eqs. (10) to (12) we find that

$$g(t) = 0 \quad \text{for all} \quad t_0 \leq t \leq t_f$$

(15)

$$u_{c}(t) = 0 \quad \text{for all} \quad t_0 \leq t \leq t_f$$

(16)

* A good example is to point my gun to shoot down a maneuvering enemy aircraft. The future position of the aircraft, the reference input, is governed by the actions of the enemy pilot which clearly cannot be predicted.
and the optimal control reduces to

\[ u(t) = -G(t)x(t) \]  

(17)

We thus conclude that the state feedback term in the optimal control is necessary to regulate the output because of the specific current energy stored in the system reflected in the value of the state \( x(t) \); the detailed tracking of any specific reference input is reflected in \( u_c(t) \).

1.6 The Equation for the Minimum Cost

If the present state of the system is \( x(t) \), then the minimum value of the cost-to-go, \( J^*(t) \), i.e.

\[ J(t) = \frac{1}{2} \Psi(t, t_f) e(t) + \frac{1}{2} \int_t^{t_f} [e'(\tau) Q(\tau) e(\tau) + u'(\tau) R(\tau) u(\tau)] d\tau \]  

(18)

is given by

\[ J^*(t) = \frac{1}{2} x'(t) K(t) x(t) + q'(t) x(t) + a(t) \]  

(19)

where the scalar \( a(t) \) satisfies the differential equation

\[ \dot{a}(t) = -\frac{1}{2} r'(t) Q(t) x(t) - q'(t) B(t) R^{-1}(t) B'(t) g(t) \]  

(20)

with the terminal time boundary condition

\[ a(t_f) = r'(t_f) K(t_f) x(t_f) \]  

(21)
2. The Deterministic Disturbance Rejection Problem

In this section we shall examine the LQ tracking problem considered in Section 1, but in the presence of disturbances which appear in an additive manner in the state dynamics of the open-loop plant.

2.1 Problem Formulation

Let us suppose that the state dynamics are described by the following vector differential equation

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + L(t)w(t) \]  \hspace{1cm} (22)

with \( x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m, \ w(t) \in \mathbb{R}^q \). The vector-valued time-function \( w(t), \ t_0 \leq t \leq t_f \) represents a disturbance input vector; the matrix \( L(t) \) "spreads" the disturbance energy in the system dynamics and captures the specific way that the disturbance input \( w(t) \) changes the state vector derivative \( \dot{x}(t) \).

The output vector \( y(t) \in \mathbb{R}^p \) is given by

\[ y(t) = C(t)x(t) \]  \hspace{1cm} (23)

The reference input \( \bar{x}(t), \ \bar{x}(t) \in \mathbb{R}^p \), is assumed known for all \( t \), \( t_0 \leq t \leq t_f \). The error signal is defined as in Section 1.1.

\[ e(t) = \bar{x}(t) - y(t) \]  \hspace{1cm} (24)

Remark: The mathematical formulation of the problem assumes that the present instantaneous value of the disturbance \( w(t) \) does not change* the instantaneous value output \( y(t) \), and hence of the tracking error \( e(t) \). The reason is that \( w(t) \) is integrated through the system dynamics first via eq. (22). This represents the most common situation

* The modeling of disturbances that directly and instantaneously influence the output \( y(t) \) should be modeled as

\[ y(t) = C(t)x(t) + \nu(t) \]

where \( \nu(t) \) is the output disturbance. One can formulate and solve the appropriate LQ problem; this will not be done here.
in physical systems. For example, suppose that $w(t)$ represents the forces and torques generated by wind disturbances upon an airplane; these are integrated at least once before they influence velocities (rates) or positions.

The quadratic cost functional that we shall use is the same as that used in Section 1.1, given by eqs. (4) and (5). The use of that cost functional implies that we still want to keep the tracking error $e(t)$ small in the presence of the state disturbance $w(t)$.

2.2 The Structure of the Optimal Control

It turns out that the optimal control $u(t)$ is given by

$$u(t) = -G(t)x(t) + u_c(t)$$  \hspace{1cm} (25)

which involves state variable feedback and a correction term $u_c(t)$.

As we shall see the vector $u_c(t)$ will depend

(a) on the specific reference input $r(t)$ and, in particular, it will depend at time $t$ on the future value of the reference input $r(\tau)$, $t \leq \tau \leq t_f$, hence having an anticipatory effect, and

(b) on the specific state disturbance $w(t)$ and, in particular, it will depend at time $t$ on the future value of the state disturbance $w(\tau)$, $t \leq \tau \leq t_f$, hence exhibiting another anticipatory effect.

2.3 Calculation of the Control Gain Matrix $G(t)$

The $m \times n$ control gain matrix $G(t)$ in eq. (25) does not depend on the specific reference input $r(t)$ and the specific disturbance $w(t)$. The matrix $G(t)$ is still defined by eqs. (7), (8) and (9).
2.4 **Calculation of the Control Correction** $u_c(t)$

The control correction vector $u_c(t)$ is given by the equation

$$u_c(t) = R^{-1}(t)B(t)h(t)$$  \hspace{1cm} (25)

where $h(t) \in \mathbb{R}^n$ is the solution vector of the linear vector differential equation

$$\dot{h}(t) = -[A(t) - B(t)G(t)]h(t) - C'(t)Q(t)x(t)$$

$$+ K(t)L(t)w(t)$$ \hspace{1cm} (27)

subject to the boundary condition at the terminal time $t_f$

$$h(t_f) = C'(t_f)x(t_f)$$ \hspace{1cm} (28)

The differential equation (27) must be solved backward in time. The reference input $r(t)$ and the disturbance $w(t)$ both act as "forcing functions" in the vector differential equation for $h(t)$. Thus the present value, at time $t$, of $h(t)$ and hence $u_c(t)$ depend on the future values of $r(\tau)$ and $w(\tau)$, $t \leq \tau \leq t_f$. The reader should also compare eqs. (26) to (28) to eqs. (10) to (12). Note that the homogeneous part for $g(t)$ and $h(t)$ are identical; only their forcing functions are different.

2.5 **The Output Regulator Problem in the Presence of Disturbances**

By setting $r(t) = 0$, we can obtain the solution to the output regulator problem in the presence of disturbances. The optimal control is given by

$$u(t) = -G(t)x(t) + u_c(t)$$ \hspace{1cm} (29)

where $G(t)$ is still defined by eqs. (7), (8), and (9), $u_c(t)$ is still given by

$$u_c(t) = R^{-1}(t)B(t)h(t)$$ \hspace{1cm} (30)
where

$$h(t) = -[A(t) - B(t)G(t)]' \ h(t) + K(t)L(t)\eta(t)$$  \hspace{1cm} (31)

with the boundary condition

$$h(t_f) = 0$$  \hspace{1cm} (32)
STEADY STATE BEHAVIOR OF THE RICCATI DIFFERENTIAL EQUATION FOR TIME-INVARIANT SYSTEMS AND OF THE CLOSED-LOOP SYSTEM

0. SUMMARY

The purpose of this note is to discuss properties of LQ regulators for time-invariant systems as the terminal time $t_f$ approaches infinity. In this manner we shall demonstrate how the algebraic Riccati equation (ARE) is obtained from the matrix Riccati differential equation.

1. Problem Formulation

1.1 Open-Loop Dynamics

Suppose that the open-loop system is linear and time-invariant. Let the state dynamics be given by

$$\dot{x}(t) = A \, x(t) + B \, u(t)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $A$, $B$ being constant matrices of appropriate dimensions.

Let us suppose that we are interested in the regulation of the following output vector $y(t)$ about the desired equilibrium value 0

$$y(t) = C \, x(t)$$

with $y(t) \in \mathbb{R}^p$ and $C$ time-invariant.
1.2 Quadratic Cost Functional

Let \( t_0 = 0 \) denote the initial time and let \( t_f \) denote the fixed terminal time. For the time being we assume \( t_f < \infty \). Consider the quadratic cost function

\[
J = \int_0^{t_f} \left[ y'(t)y(t)+u'(t)R_u(t) \right] dt \tag{3}
\]

where

\[
R = R' > 0 \quad ; \quad R = \text{constant matrix} \tag{4}
\]

By substituting eq. (2) into (3) and defining

\[
Q = C'C \quad ; \quad Q = \text{constant matrix} \tag{5}
\]

the quadratic cost function (3) can be written as

\[
J = \int_0^{t_f} \left[ x'(t)Qx(t)+u'(t)R_u(t) \right] dt \tag{6}
\]

Note that in view of eq. (5) we have

\[
Q = Q' > 0 \tag{7}
\]

The optimization problem is to find \( u(t) \), \( 0 \leq t \leq t_f \)
that minimizes the cost \( J \) subject to the state dynamic constraints (1).

1.3 Time-Varying Solution

The posed LQ problem is a special case of the time-varying LQ problem. Hence we know that the optimal control is generated by

\[
u(t) = -G(t)x(t) \tag{8}
\]

with

\[
G(t) = R^{-1}B'K(t) \tag{9}
\]
where $K(t) = K'(t)$ is the unique, at least positive semidefinite solution matrix of the matrix Riccati differential equation

$$\dot{K}(t) = -K(t)A - A'K(t) - Q + K(t)BR^{-1}B'K(t)$$  \hspace{1cm} (10)

subject to the boundary condition

$$K(t_f) = 0$$  \hspace{1cm} (11)

Remark: Our only assumption thus far is that the matrices $A, B, Q,$ and $R$ are constant; no controllability or observability assumptions have been made. Note that the control matrix $G(t)$ remains time-varying, because $K(t)$ is time-varying.

2. Asymptotic Behavior as $t_f \to \infty$

2.1 Motivation

The basic idea is to examine the behavior of the matrix solution of (10) and (11) as $t_f \to \infty$, i.e. to study

$$\lim_{t_f \to \infty} K(t)$$  \hspace{1cm} (12)

Because the matrix Riccati equation (10) is nonlinear we have to be careful about the existence and uniqueness of its solutions. We must worry about the possible existence of a finite escape time, a phenomenon illustrated by Fig. 1. The existence and uniqueness of the matrix (12), as well as other desirable properties involve many intricate proofs that are beyond the scope of these notes.

2.2 Existence and Uniqueness Results

We present two theorems that summarize the desired results.

Fig 1. Illustration of Finite Escape Time Solution.
**Theorem 2.1** Consider the matrix Riccati differential equation (10) and (11). Let \( Q \) be

\[
Q = C'C
\]  

(13)

Further make the assumptions:

\[ [A, B] \text{ is controllable} \]  

(14)

\[ [A, C] \text{ is observable} \]  

(15)

Then, the following statements are true

1. The limiting solution matrix (12) exists is unique and it is constant, i.e.

\[
\lim_{t \to \infty} K(t) = K = \text{constant}
\]  

(16)

2. The matrix \( K \) is symmetric and positive definite, i.e.

\[
K = K' > 0
\]  

(17)

3. The matrix \( K \) is the unique positive definite solution matrix of the ARE

\[
0 = -KA + A'K - Q + KBK^{-1}B'K
\]  

(18)

4. The unique optimal control generated by constant state variable feedback gains

\[
u(t) = -Gx(t)
\]  

(19)

with

\[
G = R^{-1}BK
\]  

(20)
yields an exponentially stable closed-loop system
\[ \dot{x}(t) = (A - B G)x(t) \]  
\[ y(t) = C x(t) \]  
i.e.
\[ \text{Re } \lambda_j [A - B G] < 0 \]  
\[ \text{(24)} \]

(5) The optimal cost-to-go at time \( t \) from state \( x(t) \), i.e.
\[ J = \int_t^\infty [x'(\tau) Q x(\tau) + u'(\tau) R u(\tau)] d\tau \]  
\[ \text{(25)} \]
is given by
\[ J = x'(t) K x(t) > 0 \text{ for all } x(t) \neq 0 \]  
\[ \text{(26)} \]

Remark 1: The assumptions of controllability and observability, given by eqs. (14) and (15), represent sufficient conditions for the statement of the theorem. Many of the conclusions will hold if the controllability and observability assumptions are suitably relaxed. This will be done in Theorem 2.2 below.

Remark 2: One of the most important statements of the theorem deals with the algebraic Riccati equation (ARE) of eq. (18), i.e.
\[ 0 = -K A - A' K - Q + K B R^{-1} B' K \]  
\[ \text{(27)} \]
For any matrices \( A \) and \( B \), \( Q = Q' > 0 \), \( R = R' > 0 \) the ARE (27) represents a set of simultaneous quadratic equations in the elements \( K_{ij} \) of \( K \). Depending on the dimension of the system \( n \), there may exist hundreds or thousands of solution matrices that satisfy (27); these arise from the complicated + relationships inherent in algebraic quadratic equations. What Theorem 2.1 guarantees is that, if \( [A, B] \)
is controllable and $[A,C]$ is observable, with $Q = C'C$, out of the thousands of possible solution matrices, there is one and only one symmetric and positive definite one that defined the optimal control law (19) and (20) and leads to the guaranteed closed-loop stability property (24) of the closed-loop system (21),(22) designed by the LQR method.

Remark 3: From an intuitive point of view the structure of the ARE (18) is not surprising. Let us examine the matrix Riccati differential equation (10), i.e.

$$
\dot{K}(t) = -K(t)A - A'K(t) - Q + K(t)BR^{-1}B'K(t)
$$

If indeed

$$
\lim_{t_f \to \infty} K(t) \to K = \text{constant}
$$

then

$$
\lim_{t_f \to \infty} \dot{K}(t) \to 0
$$

so that the ARE (18) or (27) is simply the limit of the differential equation (28) under facts (29) and (30).

Remark 4: The strict exponential stability of the closed-loop system is also not surprising from an intuitive viewpoint. Let us write the cost functional (25) as

$$
J = \int_{t}^{\infty} [\chi'(\tau)\chi(\tau) + u'(\tau)R_u(\tau)]d\tau
$$

From eqs. (21) and (22) the output $\chi(t)$ of the closed-loop system will be

$$
\chi(t) = C e^{[A-BG](t-t)}x(t)
$$
and in view of eq. (19) the closed loop control will be

\[ u(t) = -G \mathbf{e}^{(A-BG)(T-t)} \mathbf{x}(t) \]  

(33)

If the closed loop system were not stable and \([A, C]\) is observable, then at least one component of \(y(t)\) and/or \(u(t)\) would grow exponentially in time; therefore in (31) we would integrate over an infinite time interval positive growing exponentials hence I would "blow up" to \(\infty\). Infinity is hardly the minimum of anything! Since we know that the controllability of \([A, B]\) together with linear state variable feedback of the form (19) can stabilize any given system, we know that we can find gain matrices \(G\) that make the cost (31) finite, and hence the minimum cost must be finite! Hence, we have reached a contradiction; it follows under our assumptions that the LQ optimization problem must lead to a strictly stable (exponentially stable) closed-loop system.

The above remarks give us a clue on how to relax the assumption of \([A, B]\) controllability and \([A, C]\) observability. We want to relax these assumptions in such a way so that:

1. the LQ closed-loop system is still guaranteed to be asymptotically stable
2. the LQ dynamic optimization leads to a finite cost.

Let us see how we can relax the \([A, B]\) controllability assumption. Suppose that the system

\[ \dot{x}(t) = A \mathbf{x}(t) + B \mathbf{u}(t) \]  

(34)

is assumed stabilizable, which means that all unstable modes of the open-loop system (34) are controllable; one or more stable modes of (34) may not be controllable.* Stabilizability implies that there exists at least one control gain matrix \(G\) such that the control law

\[ u(t) = -G \mathbf{x}(t) \]  

(35)

leads to an exponentially stable closed-loop system

\[ \dot{x}(t) = [A-BG] \mathbf{x}(t) \]  

(36)

* A simple example of a stabilizable system is

\[ \dot{x}_1(t) = -x_1(t) \]
\[ \dot{x}_2(t) = x_2(t) + u(t) \]

Clearly the stable mode \(x_1(t)\) is not controllable but the unstable mode \(x_2(t)\) is controllable. Clearly the control law

\[ u(t) = -gx_2(t); \ g>1 \]

leads to a strictly stable closed loop system.
If the system (34) is stabilizable, and the control law (35) leads to a strictly stable closed loop system, then the control \( u(t) \) will consist of decaying exponentials and for \( R = R' > 0 \)

\[
\int_{0}^{\infty} u'(t) R u(t) dt = \text{finite and nonnegative} \quad (37)
\]

Next let us see how we can relax the \([A, C]\) observability assumption. Let us suppose that the open-loop system

\[
\dot{x}(t) = A x(t) + B u(t); \quad y(t) = C x(t) \quad (38)
\]

is assumed to be detectable, which means that all unstable modes of the open-loop system are observable; one or more stable modes of the open-loop system may not be observable. *

Notice that if the open-loop system (38) is detectable, then if \( u(t) = 0 \), the quantity

\[
\int_{0}^{\infty} y'(t) y(t) dt \quad (39)
\]

will be finite if and only if the open-loop system has unstable modes. The integral (39) will blow-up (for \( u(t) = 0 \)) if the detectable system has unstable modes.

It should be now clear that if the system (38) is both stabilizable and detectable, then there exists at least one \( C \) such that the control law (35) will make the cost functional

\[
J = \int_{0}^{\infty} [y'(t) y(t) + u'(t) R u(t)] dt \quad (40)
\]

* A simple example of a detectable system is

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) \\
\dot{x}_2(t) &= x_2(t) \\
y(t) &= x_2(t)
\end{align*}
\]

Clearly the stable mode \( x_1(t) \) is not observable, but the unstable mode \( x_2(t) \) is observable.
both finite and nonnegative. Thus, the LQ dynamic optimization problem will make sense.

The above discussion is summarized in the following theorem.

Theorem 2.2: Consider the matrix Riccati differential equation (10) and (11). Let Q be

\[ Q = C'C \]  

Further, make the assumptions that

\[ [A,B] \text{ is stabilizable} \]

\[ [A,C] \text{ is detectable} \]  

Then the following statements are true:

\( \mathbf{1} \) The limiting solution matrix (12) exists, is unique and it is constant, i.e.

\[ \lim_{t \to \infty} K(t) = K = \text{constant} \]  

\( \mathbf{2} \) The matrix K is symmetric and at least positive semidefinite, i.e.

\[ K = K' > 0 \]  

\( \mathbf{3} \) The matrix K is the unique positive semidefinite solution matrix of the ARE

\[ 0 = -KA - A'K - Q + KB R^{-1} B'K \]  

\( \mathbf{4} \) The unique optimal control generated by constant state variable feedback gains

\[ u(t) = -G x(t) \]  

with

\[ G = R^{-1} B'K \]
yields an exponentially stable closed-loop system

\[
\dot{x}(t) = [A-BG]x(t)
\]

\[
y(t) = Cx(t)
\]

i.e.

\[
\text{Re } \lambda \succ 0 \text{ for all } x \neq 0
\]

(50)

(5) The optimal cost-to-go at time t from state \(x(t)\), i.e.

\[
J = \int_{t}^{\infty} [x'(t)Qx(t)+u'(t)Ru(t)]dt
\]

is given by

\[
J = \langle x'(t)x(t) \rangle \geq 0 \text{ for all } x(t) \neq 0
\]

(52)

A "Silly Example"

Consider the system

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + u(t) \\
\dot{x}_2(t) &= -2x_2(t) \\
y(t) &= x_2(t)
\end{align*}
\]

(53)

Clearly the system (53) is both stabilizable and detectable.

The cost functional \(J\) is

\[
J = \int_{0}^{\infty} [y^2(t)+u^2(t)]dt
\]

(54)
The optimal solution should be obvious. The control \( u(t) \) cannot influence \( x_2(t) = y(t) \); since it costs us to use controls the optimal control must be zero.

Let us see what the calculations tell us. The matrices \( A, \ B, \ C, \) and \( R \) for this problem are

\[
A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0 \ 1], \quad R = 1
\]

(55)

The ARE is, using that \( K = K' \),

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ 1] \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 0] \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}
\]

(56)

Carrying out the matrix multiplications we have

\[
\begin{align*}
0 &= 2K_{11} + K_{12}^2 \\
0 &= 3K_{12} + K_{11}K_{12} \\
0 &= 4K_{22} - 1 + K_{12}
\end{align*}
\]

(57)

These turn out to have a unique solution

\[
K_{11} = 0, \quad K_{12} = 0, \quad K_{22} = \frac{1}{4}
\]

(58)
which imply the positive semidefinite solution matrix

\[ K = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \]  \hspace{1cm} (58)

The control gain matrix \( G \) is

\[ G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \]  \hspace{1cm} (59)

hence the optimal control is

\[ u(t) = -0 \cdot x_1(t) - 0 \cdot x_2(t) = 0 \]  \hspace{1cm} (60)

The minimum cost \( J = x'(0)Kx(0) \) is

\[ J = \frac{1}{4} x_2^2(0) \]  \hspace{1cm} (61)

Equation (61) is hardly surprising since

\[ x_2(t) = e^{-2t} x_2(0) \]  \hspace{1cm} (62)

hence

\[ \int_{0}^{\infty} x_2^2(t) dt = \int_{0}^{\infty} e^{-4t} dt \cdot x_2^2(0) = \frac{1}{4} x_2^2(0) \]  \hspace{1cm} (63)
THE TIME-VARYING LINEAR QUADRATIC REGULATOR (LQR) PROBLEM

0. SUMMARY

In this note we present a variational proof of the Linear-Quadratic-Regulator (LQR) problem. The results obtained for the time-varying problem can then be specialized for the time-invariant case through some additional assumptions.

1. Problem Statement

In this section we present the statement of the problem. We remark that the LQR problem is a specific example of deterministic dynamic optimization problems for which a rich theory and algorithmic solution methods have evolved since the early 1960's.

1.1 Open-Loop State Dynamics

Consider a linear time-varying dynamic system described in the time-domain by the vector differential equation

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t); \quad x(t_0) = x_0
\]  

(1)

We assume that \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \). The initial \( x_0 \) at the initial time to is fixed but arbitrary.

We remark that no controllability or observability assumptions are made.
1.2 The Cost Functional

Suppose that the system (1) is initially excited, and that the net result of this excitation is reflected in the initial state vector \( x_0 \).

The control objective is to select the control vector \( u(t) \), during a finite time-interval \( t_0 \leq t \leq t_f \), where \( t_f \) is called the final time (or planning horizon), so that the state vector \( x(t) \) is near zero. Thus, \( x(t)=0 \) is the desired equilibrium state for the system (1).

If the system (1) is controllable, then it is obvious that one could drive \( x_0 \) to 0 in an arbitrarily short period of time; a control impulse at \( t=t_0 \) could accomplish this objective.

From an engineering point of view very large control signals are undesirable because

- (a) they can saturate actuators
- (b) if implemented in a feedback form, large control magnitudes require large control gains and yield high-bandwidth designs that may excite unmodeled dynamics.

For the above reasons we must balance our desire to drive \( x_0 \) toward 0 by requiring that this does not lead to excessively large control signals.

One way of doing this is to form a cost functional, \( J \), of the following form:

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \left[ x'(t)F x(t) + u'(t)R(t)u(t) \right] dt
\]

where \( F \) and \( Q(t) \) are symmetric positive semi-definite matrices, and \( R(t) \) is a symmetric positive definite matrix.

*The \( \frac{1}{2} \) factors in eq. (2) are introduced for the purpose of convenience.*
The term in the cost function (2)

\[ u'(t)R(t)u(t) > 0 \quad \text{for} \quad u(t) \neq 0 \quad (3) \]

represents a quadratic penalty that helps the designer to keep the magnitude of \( u(t) \) "small". The matrix \( R(t) \) is often called the control weighting matrix. The quadratic nature of (3) yields a larger penalty for large values of \( u(t) \) as compared with small ones. It is precisely the selection of the control weighting matrix \( R(t) \) that allows the control system designer to selectively control the loop-gains and the bandwidths of each control channel.

The term in the cost functional (2)

\[ x'(t)Q(t)x(t) > 0 \quad \text{for} \quad x(t) \neq 0 \quad (4) \]

generates a penalty when all (or some) of the states are different from their desired equilibrium value \( 0 \). The selection of the state weighting matrix \( Q(t) \) is the means by which the control system designer communicates to the mathematics the relative importance of individual state variable deviations, i.e. which errors are more bothersome and to what degree.

The term in the cost function (2)

\[ x'(t_f)F x(t_f) > 0 \quad \text{for} \quad x(t_f) \neq 0 \quad (5) \]

is called the terminal state penalty. There are many finite-time problems* in which the deviations of the final values of some state variables from their desired equilibrium values are especially important. The selection of the terminal state weighting matrix \( F \) allows the designer to communicate this desired behavior of the terminal state \( x(t_f) \) to the mathematics.

It should be evident that for the same value of initial state \( x_0' \) different controls \( u(t), t_0 \leq t \leq t_f \), would yield different positive values of the cost \( J \) in eq. (2). Obviously, the values of \( x(t), t_0 \leq t \leq t_f \) are constrained to be the solutions of the state dynamics differential equation (1).

*Such as missile interception problems, rendezvous problems, satellite orbit insertion problems, etc.
The basic idea of the dynamic optimization problem, the LQR problem, is to find the vector control function $u(t)$ for $t_0 \leq t \leq t_f$ which will make the cost functional (2) as small as possible, subject to the constraint that $x(t)$ and $u(t)$ are related by eq. (1).

2. Precise Statement of the Solution to the LQR Problem

Given the quadratic cost functional

$$J = \frac{1}{2} x'(t_f)F_x x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)] dt$$

with

$$F = F' > 0; \quad Q(t) = Q'(t) > 0; \quad R(t) = R'(t) > 0.$$  

and the dynamic constraint

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \quad x(t_0) = x_0$$

Then there exists a unique optimal control $u(t)$, $t_0 \leq t \leq t_f$, that minimizes the cost (6) subject to the constraint (8). Furthermore, the optimal control can be generated by full state variable feedback according to the control law

$$u(t) = -G(t)x(t)$$

with the $m \times n$ time-varying control gain matrix $G(t)$ given by

$$G(t) = R^{-1}(t)B'(t)K(t)$$

where the $n \times n$ matrix $K(t) = K'(t)$, $t_0 \leq t \leq t_f$ is the unique, at least positive semidefinite, backwards-in-time solution matrix of the following matrix Riccati differential equation

$$\frac{d}{dt} K(t) = -K(t)A(t) - A'(t)K(t) - Q(t)$$

$$+ K(t)B(t)R^{-1}(t)B'(t)K(t)$$
subject to the boundary condition at the terminal time $t_f$

$$K(t_f) = F$$  \hspace{1cm} (12)

Furthermore, the minimum value of the cost $J$ in eq. (6) is given by the quadratic form

$$J = \frac{1}{2} x' \begin{bmatrix} K(t_0) & x_0 \end{bmatrix} \tag{13}$$

3. Remarks

Before we present the variational proof of the LQR problem it is important to stress some of its features.

First, let us consider the way the optimal control is constructed from eqs. (9) and (10)

$$u(t) = -G(t)x(t) = -R^{-1}(t)B'(t)K(t)x(t)$$  \hspace{1cm} (14)

Think of $t$ as the present time and $t_f$ as the future time. The optimal control depends on the present state $x(t)$ which summarizes what has happened in the past. However, the Riccati matrix $K(t)$, and hence the gain matrix $G(t)$, will depend on the future values of the parameters of the dynamic system, i.e. the matrices $A(t), B(t)$ for $t \leq t \leq t_f$ (15)

and the future values of the weighting matrices $Q(t), R(t)$ for $t \leq t \leq t_f$ (16)

in the quadratic performance index $J$, as well as the terminal state weighting matrix $F$.

The reason is that the Riccati matrix differential equation (11) is integrated backward in time* starting at the boundary condition (12). Hence, at time $t$ the change in the future system dynamics (i.e. the

*One needs a digital computer to numerically calculate $K(t)$.\]
matrices $A(t)$ and $B(t), t \leq t \leq t_f$ and the future structure of the cost functional (i.e. the matrices $Q(t)$ and $R(t), t \leq t \leq t_f$) are of importance.

The above properties are fundamental to any dynamic optimal control problem; the optimal control now must take into account what is optimized in the future; it cannot change the past.

4. Variational Proof of the LQR Problem

The philosophy behind all variational proofs is to assume existence of the optimal control and then deduce its properties. Most variational proofs generate necessary conditions for optimality; with a little more work one can establish sufficient conditions as well.

4.1 Existence of Optimal Control

As long as the matrices $A(t), B(t), Q(t)$, and $R(t)$ are piecewise-continuous with a finite number of discontinuities in the time-interval $t_0 \leq t \leq t_f$, and they are bounded, then the optimal control $u^*(t)$ exists.

It follows that there exists an optimal state trajectory, denoted by $x^*(t)$, which must satisfy the state dynamics, i.e.

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t); \quad x^*(t_0) = x_0$$  \hspace{1cm} (17)

Then the optimal cost, denoted by $J(u^*)$ is given by

$$J(u^*) = \frac{1}{2} x^*(t_f) F x^*(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x^*(t)Q(t)x^*(t) + u^*(t)R(t)u^*(t)] dt$$  \hspace{1cm} (18)

4.2 Perturbations

Let $\epsilon$ be an arbitrary positive or negative scalar, and let

$\tilde{u}(t) \in R^m$ be an arbitrary vector-valued time function for $t_0 \leq t \leq t_f$.

Consider the control vector $u(t)$ defined by
\[ u(t) = u^*(t) + \epsilon \tilde{u}(t) \]  

(19)

We next compute the state \( x(t) \) generated by the control (19) from

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + \epsilon \tilde{B}(t)\tilde{u}(t); \quad x(t_0) = x_0 \]  

(20)

Denote the solution of (20) as

\[ x(t) = x^*(t) + \epsilon \tilde{x}(t) \]  

(21)

From eqs. (17) and (20) we readily deduce that the state perturbation \( \tilde{x}(t) \) is related to the control perturbation \( \tilde{u}(t) \) by

\[ \frac{d}{dt} \tilde{x}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t); \quad \tilde{x}(t_0) = 0 \]  

(22)

4.3 The Costate Vector

In dynamic optimization problems we are interested in minimizing a cost functional subject to differential equation constraints. In static optimization problems one wants to minimize a cost subject, say, to certain equality constraints. In static problems, one uses Lagrange multipliers to adjoin the equality constraints to the cost that is to be minimized. An analogous method is used in dynamic optimization problems in which a time-varying vector, called the costate, is used to adjoin the differential equation constraints to the cost functional to be minimized.

Let us write the differential equations (1) as follows

\[ A(t)x(t) + B(t)u(t) - \dot{x}(t) = 0 \]  

(23)

Let \( p(t) \in R^n, \quad t_0 \leq t \leq t_f \) denote the costate vector; at this point it is an arbitrary time-varying vector. Next, compute the scalar product of \( p(t) \) with the \( 0 \) vector defined by eq. (23).
\[ p'(t)[A(t)x(t)+B(t)u(t)-\dot{x}(t)] = 0 \]  

(24)

Then, for any \( u(t) \) the cost \( J(u) \) given by eq. (2) does not change by adding 0 to the integrant. Hence

\[
J(u) = \frac{1}{2} x'(t_f) P x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(t)x(t)+u'(t)R(t)u(t)] dt
\]

\[ + \int_{t_0}^{t_f} p'(t)A(t)x(t)dt + \int_{t_0}^{t_f} p'(t)B(t)u(t)dt - \int_{t_0}^{t_f} p'(t)\dot{x}(t)dt \]  

(25)

We next evaluate the last term of eq. (25) using the rules of integration by parts

\[
\int_{t_0}^{t_f} p'(t)x(t)dt = p'(t_f)x(t_f) - p'(t_0)x(t_0) - \int_{t_0}^{t_f} p''(t)x(t)dt
\]  

(26)

We substitute eq. (26) into eq. (25) to obtain, after collecting terms,

\[
J(u) = \frac{1}{2} x'(t_f) P x(t_f) - p'(t_f)x(t_f) + p'(t_0)x(t_0)
\]

\[ + \int_{t_0}^{t_f} \left[ \frac{1}{2} x'(t)Q(t)x(t)+p'(t)A(t)x(t)+p'(t)\dot{x}(t) \right] dt \]

\[ + \int_{t_0}^{t_f} \left[ \frac{1}{2} u'(t)R(t)u(t)+p'(t)B(t)u(t) \right] dt \]  

(27)
We now write the minimum cost $J(u^*)$ induced by the optimal control $u^*(t)$ and the resultant optimal state trajectory $x^*(t)$. From eq. (27) we have

$$J(u^*) = \frac{1}{2} x^*(t_f)F x^*(t_f) - \int_{t_0}^{t_f} \left[ \frac{1}{2} x^*(t)^T Q(t) x^*(t) + p'(t) A(t) x^*(t) + p'(t) x^*(t) \right] dt$$

$$+ \int_{t_0}^{t_f} \left[ \frac{1}{2} u^*(t)^T R(t) u^*(t) + p'(t) B(t) u^*(t) \right] dt$$

(28)

4.4 Implications of Optimality

Since by definition $u^*(t)$ and $x^*(t)$ are optimal, then

$$J(u) - J(u^*) \geq 0$$

(29)

for all $u(t)$, $t_0 \leq t \leq t_f$. This leads us to compute the cost difference $J(u) - J(u^*)$ using eqs. (27) and (28) and the perturbation equations (19) and (21); also recall that

$$x(t_0) = x^*(t_0) = x_0$$

(30)

After some algebraic manipulations (lengthy but straightforward) we find...
\[
J(u) - J(u^*) = \varepsilon \left\{ \left[ x^{**}(t_f) F - p'(t_f) \right] \bar{x}(t_f) \right\} \\
+ \varepsilon \int_{t_0}^{t_f} \left[ x^{**}(t) Q(t) + p'(t) A(t) + \dot{x}(t) \right] \bar{x}(t) dt \\
+ \varepsilon \int_{t_0}^{t_f} \left[ u^{**}(t) R(t) + p'(t) B(t) \right] \bar{u}(t) dt \\
+ \varepsilon^2 \frac{1}{2} \int_{t_0}^{t_f} \dot{x}(t) \bar{x}(t) dt \\
+ \varepsilon^2 \frac{1}{2} \int_{t_0}^{t_f} \ddot{u}(t) R(t) \bar{u}(t) dt \geq 0 
\]

The next step is to let $\varepsilon \to 0$; in this limiting case the terms multiplied by $\varepsilon^2$ in eq. (31) can be neglected as compared to the terms multiplied by $\varepsilon$. After we neglect the $\varepsilon^2$ terms we examine the ratio

\[
\frac{1}{\varepsilon} \left[ J(u) - J(u^*) \right] > 0 \quad \text{if} \quad \varepsilon > 0 \\
\frac{1}{\varepsilon} \left[ J(u) - J(u^*) \right] \leq 0 \quad \text{if} \quad \varepsilon < 0 
\]

The only way that both the inequalities in (32) can be true is to have

\[
J(u) - J(u^*) = 0 
\]
This necessarily implies the equality

\[ 0 = [x^*(t_f) F - p^*(t_f)] \tilde{x}(t_f) \]
\[ + \int_{t_0}^{t_f} [x''(t) Q(t) + p'(t) \dot{A}(t) + \dot{p}'(t)] \tilde{x}(t) \, dt \]
\[ + \int_{t_0}^{t_f} [u''(t) R(t) + p'(t) \dot{R}(t)] \tilde{u}(t) \, dt \]  
(34)

4.5 The Costate Dynamics

Up to now the costate vector \( p(t) \) was completely arbitrary. The idea is that we can select a particular costate vector, associated with \( u^*(t) \) and \( x^*(t) \), so as to satisfy the equality (34). Note that the state perturbation \( \tilde{x}(t) \) and \( \tilde{x}(t_f) \) are not arbitrary but depend upon the control perturbation \( \tilde{u}(t) \) which is arbitrary—according to eq. (22). Thus, the specific costate vector, which we shall denote by \( p^*(t) \), is selected so that

\[ [x''(t_f) F - p^*(t_f)] = 0 \]

and that

\[ [x''(t) Q(t) + p'(t) \dot{A}(t) + \dot{p}'(t)] = 0 \]  
(36)

From eq. (36) we deduce that \( p^*(t) \) must satisfy the linear vector differential equation

\[ \dot{p}^*(t) = -A'(t)p^*(t) - Q(t)x^*(t) \]  
(37)

and the boundary condition

\[ p^*(t_f) = F x^*(t_f) \]  
(38)

We remark that the costate vector \( p^*(t) \) exists and is unique for all \( t_0 < t < t_f \). In view of the existence and uniqueness properties of linear vector differential equations with bounded piecewise continuous coefficients
4.6 The Optimal Control Equation

Next we substitute eqs. (37) and (38) into eq. (34) to obtain

\[ 0 = \int_{t_0}^{t_f} [u^*(t)R(t) + p^*(t)B(t)] \dot{u}(t) \, dt \quad (39) \]

The equality (39) must hold for any \( \tilde{u}(t) \); recall that \( \tilde{u}(t) \) is completely arbitrary. For (39) to be true the integrand must be identically zero for all \( t_0 \leq t \leq t_f \) and for all \( \tilde{u}(t) \). This necessarily implies that

\[ u^*(t)R(t) + p^*(t)B(t) = 0 \quad (40) \]

Hence, the optimal control \( u^*(t) \) must necessarily be uniquely related to the costate \( p^*(t) \) by the linear algebraic equation

\[ u^*(t) = -R^{-1}(t)B'(t)p^*(t) \quad (41) \]

4.7 Sufficiency

One would obtain the equality (34) if \( u^*(t) \) maximized \( J(u) \). Thus it remains to be proved that it minimizes the quadratic cost functional. Thus, we must return to the fundamental inequality (31) and examine it closely.

We substitute eqs. (37), (38), and (41) into eq. (31) to obtain

\[ J(u) - J(u^*) = \varepsilon^2 \left\{ \frac{1}{2} \left[ \ddot{x}(t_f)R \ddot{x}(t_f) \right] + \int_{t_0}^{t_f} [\dddot{x}(t)Q(t)\ddot{x}(t) + \dddot{u}(t)R(t)\dot{u}(t)] \, dt \right\} \geq 0 \quad (42) \]
The above inequality must be true for arbitrary control perturbations \( \bar{u}(t) \) and induced state perturbations \( \bar{x}(t) \) from eq. (22). Thus we ask the question whether (42) is true. The answer is YES because \( \varepsilon > 0 \) and because of our assumptions that \( P \) and \( Q(t) \) are positive semidefinite matrices and that \( R(t) \) is a positive definite matrix.

This establishes that indeed \( u^*(t) \) is a minimizing control.

4.8 Partial Summary

Let us summarize what we have proved up to now. We have shown for the LQR problem that there exists a unique optimal control \( u^*(t) \), a corresponding optimal state trajectory \( x^*(t) \) and a corresponding costate \( p^*(t) \) for all \( 0 \leq t \leq T \) such that the following equations hold.

State and Costate Dynamics

The time function \( x^*(t) \), \( p^*(t) \), and \( u^*(t) \) must satisfy the following differential equations

\[
\begin{align*}
\dot{x}^*(t) &= A(t)x^*(t) + B(t)u^*(t) \\
p^*(t) &= -Q(t)x^*(t) - A'^*(t)p^*(t)
\end{align*}
\]

Boundary Conditions

At the initial time \( t_0 \):

\[
x^*(t_0) = x_0
\]

At the terminal time \( t_f \):

\[
p^*(t_f) = F x^*(t_f)
\]
Relation of Optimal Control

\[ u^*(t) = -R^{-1}(t)B'(t)\pi^*(t) \] (47)

Although not obvious unless one solves some numerical exercises, equations (43) through (47) can be used to find the numerical solution to any LQR problem.

We remark that the development so far does not provide the solution to the LQR problem as stated in Section 2. To accomplish this objective one must further manipulate the necessary and sufficient conditions for optimality which are eqs. (43) to (47). This will be done in the sequel.

4.9 The LQR Two-Point Boundary Value Problem

We shall now commence our manipulation of the necessary and sufficient conditions to eventually arrive at the feedback form of the LQR problem and the matrix Riccati differential equation.

We start by substituting eq. (47) into eq. (43) so as to derive a set of coupled linear vector differential equations for the optimal state \( x^*(t) \) and costate \( \pi^*(t) \). These are

\[
\begin{bmatrix}
\dot{x}^*(t) \\
\dot{\pi}^*(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & -B(t)R^{-1}(t)B'(t) \\
-\Omega(t) & -A'(t)
\end{bmatrix}
\begin{bmatrix}
x^*(t) \\
\pi^*(t)
\end{bmatrix}
\] (48)

These 2n differential equations are often called the Hamiltonian system. Since eq. (48) is a linear vector differential equation it has a 2nx2n transition matrix \( \Phi(t;\tau) \). We decompose \( \Phi(t;\tau) \) into its nxn blocks as follows

\[
\Phi(t;\tau) =
\begin{bmatrix}
\Phi_{11}(t;\tau) & \Phi_{12}(t;\tau) \\
\Phi_{21}(t;\tau) & \Phi_{22}(t;\tau)
\end{bmatrix}
\] (49)
Using solution concepts from linear system theory we can write the solution to eq. (48) as follows:

\[ x_\ast(t_f) = \Phi_{11}(t_f;t)x_\ast(t) + \Phi_{12}(t_f;t)p_\ast(t) \]  
\[ p_\ast(t_f) = \Phi_{21}(t_f;t)x_\ast(t) + \Phi_{22}(t_f;t)p_\ast(t) \]  

Using the boundary condition (46), i.e.

\[ p_\ast(t_f) = \mathcal{F}x_\ast(t_f) \]  

together with eqs. (50) and (51) we obtain the vector equality

\[ \Phi_{21}(t_f;t)x_\ast(t) + \Phi_{22}(t_f;t)p_\ast(t) \]
\[ = \mathcal{F}\Phi_{11}(t_f;t)x_\ast(t) + \mathcal{F}\Phi_{12}(t_f;t)p_\ast(t) \]  

Next we solve for \( p_\ast(t) \) in terms of \( x_\ast(t) \) to find

\[ p_\ast(t) = K(t)x_\ast(t) \]  

where the \( nxn \) matrix \( K(t) \) is given by

\[ K(t) = [\Phi_{22}(t_f;t) - \mathcal{F}\Phi_{12}(t_f;t)]^{-1} \cdot [\mathcal{F}\Phi_{11}(t_f;t) - \Phi_{21}(t_f;t)] \]  

Substituting eq. (54) into eq. (47) we deduce that the optimal control \( u_\ast(t) \) can be generated via linear feedback from the optimal state \( x_\ast(t) \) according to the equation

\[ u_\ast(t) = -K^{-1}(t)B'(t)K(t)x_\ast(t) \]  

where \( K(t) \) is given by eq. (55).
4.10 Remarks

One could stop at this point since a complete algorithm has been developed to generate the optimal control using time-varying state variable feedback from the state

\[ u^*(t) = -G(t)x^*(t) \]  \hspace{1cm} (57)

where the control gain matrix \( G(t) \) is defined by

\[ G(t) = R^{-1}(t)B'(t)K(t) \]  \hspace{1cm} (58)

and the matrix \( K(t) \) is given by eq. (55). Indeed in the early 1960's, the so-called ASP program calculated \( K(t) \) according to eq. (55). However, there can be severe numerical problems in the digital computer evaluation of eq. (55) including the complexity of computing the inverse of an \( nxn \) time-varying matrix. This motivated the development of an alternate algorithm for calculating \( K(t) \); happily, this alternate algorithm, the matrix Riccati differential equation provided additional analytic insight.*

Before we proceed, it is important to notice that the calculation of \( K(t) \) can be done off-line. Thus, the messy calculation that define the control gain matrix \( G(t) \) according to eqs. (58) and (55) do not take place in real-time.

4.11 The Matrix Riccati Differential Equation

We shall now demonstrate how the calculation of \( K(t) \) can be carried out in an alternate manner.

Our starting point is the relation (54) that relates the state and costate, i.e.

\[ p^*(t) = K(t)x^*(t) \]  \hspace{1cm} (59)

We take time derivatives to obtain

\[ \dot{p}^*(t) = \dot{K}(t)x^*(t) + K(t)\dot{x}^*(t) \]  \hspace{1cm} (60)

*It also helped to established the duality of the LQR problem to the Kalman filter algorithm for state estimation problems.
We now substitute eq. (48) into eq. (60) to eliminate \( \dot{P}^*(t) \) and \( \dot{x}^*(t) \); this yields

\[
-Q(t)x^*(t) - A'(t)P^*(t) = K(t)x^*(t)
\]
\[
+ K(t)A(t)x^*(t) - K(t)B(t)R^{-1}(t)B'(t)P^*(t)
\]

(61)

We substitute eq. (59) into eq. (61) and collect terms to obtain

\[
[K(t)+K(t)A(t)+A'(t)K(t)+Q(t)]x^*(t) = 0
\]

(62)

Note that \( x^*(t) \) is arbitrary, since the initial state \( x_0 \) is arbitrary; hence, the only way that eq. (62) can hold for all \( t_0 \leq t \leq t_f \) is that the matrix in brackets in eq. (62) must be identically zero. This in turn implies that the nxn matrix \( K(t) \) must satisfy the matrix Riccati differential equation

\[
\frac{d}{dt}K(t) = -K(t)A(t)-A'(t)K(t)-Q(t)+K(t)B(t)R^{-1}(t)B'(t)K(t)
\]

(63)

To uniquely define \( K(t) \), \( t_0 \leq t \leq t_f \), one must also specify a boundary condition. From eq. (59) we have

\[
P^*(t_f) = K(t_f)x^*(t_f)
\]

(64)

However (see eq. (62)) we have the necessary condition

\[
P^*(t_f) = P^* x^*(t_f)
\]

(65)

From eqs. (64) and (65) we deduce that the sought-for boundary condition for \( K(t) \) is given at the terminal time \( t_f \) by

\[
K(t_f) = P
\]

(66)

This completes the derivation of the Riccati equation, and established the LQR solution summarized in Section 2 of this note.

The nxn solution matrix \( K(t) \) of the Riccati differential equation (63) subject to the boundary condition (66) turns out to be symmetric, i.e.

\[
K(t) = K'(t)
\]

(67)
This property is due to the fact that $F$, $Q(t)$, and $R(t)$ are symmetric. It can be readily verified by substituting $X'(t)$ in the differential equation (63) and the boundary condition (66) and demonstrate that it is indeed a solution.

### 4.12 An Expression for the Optimal Cost

The minimum value of the optimal cost denoted by $J^* = J(u^*)$ is evidently given by

$$J^* = \frac{1}{2} x^*(t_f)F x^*(t_f)$$

$$+ \frac{1}{2} \int_{t_0}^{t_f} \left[ [x^*(t)Q(t)x^*(t)+u^*(t)R(t)u^*(t)] \right] dt$$

(67)

We know that the optimal control $u^*(t)$ is

$$u^*(t) = -R^{-1}(t)B'(t)K(t)x^*(t)$$

(68)

and that the optimal state $x^*(t)$ propagates according to the closed-loop dynamics

$$\dot{x}^*(t) = [A(t)-B(t)R^{-1}(t)B'(t)K(t)]x^*(t); \quad x^*(t_0) = x_0$$

(69)

Using eqs. (67) to (69) one can establish* that the minimum cost $J^*$ is given by the quadratic form

$$J^* = \frac{1}{2} x_0' K(t_0) x_0$$

(70)

Since $x_0 = x(t_0)$ is arbitrary, and $t_0$ is arbitrary one can readily conclude that

$$K(t) \geq 0; \quad t_0 \leq t \leq t_f$$

(71)

since $J(u) \geq 0$ for any control $u(t)$.

*The derivation is summarized in Appendix A.
APPENDIX A

DERIVATION OF OPTIMAL COST FOR LINEAR QUADRATIC REGULATORS

Given the linear system

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (A-1) \]

and the quadratic cost function

\[ J(u) = \frac{1}{2} x'(t_f)x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)] dt \quad (A-2) \]

We proved that:

1. The optimal control is given by, for all \( t \in [t_0, t_f] \),
   \[ u^*(t) = -R^{-1}(t)B'(t)K(t)x^*(t) \quad (A-3) \]

2. \( x^*(t) \) satisfies the "closed-loop" equation
   \[ \dot{x}^*(t) = [A(t) - B(t)R^{-1}(t)B'(t)K(t)]x^*(t); \quad x^*(t_0) = x_0 \quad (A-4) \]

3. The matrix \( K(t) \) is symmetric and satisfies the matrix Riccati differential equation.

\[ \dot{K}(t) = -K(t)A(t)A'(t)K(t) + K(t)B(t)R^{-1}(t)B'(t)K(t) - Q(t) \quad (A-5) \]

subject to the boundary condition, at the terminal time \( t_f \),

\[ K(t_f) = P \quad (A-6) \]
Suppose that the system is at state \( x \) at time \( t \). We shall show that the minimum cost "to go" is

\[
J^* = \frac{1}{2} x^*(t_f)\Phi_f(t_f) + \frac{1}{2} \int_t^{t_f} [x^*(τ)Q(τ)x^*(τ) + u^*(τ)R(τ)u^*(τ)]dτ = \frac{1}{2} x^*K(t)x
\]

(A-7)

To prove equation (A-7) we proceed as follows. From Eq. (A-3) we deduce that the following relation holds (by forming scalar products)

\[
u^*(τ)R(τ)u^*(τ) = x^*(τ)K(τ)B(τ)R^{-1}(τ)B'(τ)K(τ)x^*(τ)
\]

(A-8)

From Eq. (A-4) we know that the state along the optimal trajectory is

\[
x^*(τ) = \Phi(τ, t)x
\]

(A-9)

where \( \Phi(t, t_0) \) is the transition matrix associated with the "closed loop system matrix" \( [A(t) - B(t)R^{-1}(t)B'(t)K(t)] \). Thus \( \Phi(\cdot, \cdot) \) is defined by:

\[
\frac{d}{dt} \Phi(τ, t) = [A(τ) - B(τ)R^{-1}(τ)B'(τ)K(τ)] \Phi(τ, t); \quad \Phi(t, t) = I
\]

(A-10)

Substitution of Eq. (A-9) into (A-8) and (A-7) yields

\[
J^* = \frac{1}{2} x'\Phi'(t_f, t)\Phi_f(t_f) x
\]

\[
+ \frac{1}{2} x' \left[ \int_t^{t_f} \Phi'(τ, t) [Q(τ) + K(τ)B(τ)R^{-1}(τ)B'(τ)K(τ)] \Phi(τ, t) dτ \right] x
\]

(A-11)

From the Riccati equation (A-5) we have (by adding and subtracting appropriate terms)

\[
Q(τ) + K(τ)B(τ)R^{-1}(τ)B'(τ)K(τ) =
\]

\[
- A(τ)K(τ) + 2K(τ)B(τ)R^{-1}(τ)B'(τ)K(τ) - \frac{d}{dt} K(τ) =
\]

\[
- K(τ) [A(τ) - B(τ)R^{-1}(τ)B'(τ)K(τ)] - [A(τ) - B(τ)R^{-1}(τ)B'(τ)K(τ)]K(τ) - \frac{d}{dt} K(τ)
\]

(A-12)
But since Eq. (A-10) states that

\[ \frac{d}{dt} \Phi(t,t) = [A(t) - B(t) R^{-1}(t) B'(t) K(t)] \Phi(t,t) \]  

(A-13)

by taking transposes we conclude that

\[ \frac{d}{dt} \Phi'(t,t) = \Phi'(t,t) [A(t) - B(t) R^{-1}(t) B'(t) K(t)]' \]  

(A-14)

Using the properties of transition matrices we have

\[ [A(t) - B(t) R^{-1}(t) B'(t) K(t)] = \frac{d}{dt} \Phi(t,t) \Phi(t,t) \]  

(A-15)

\[ [A(t) - B(t) R^{-1}(t) B'(t) K(t)]' = \Phi'(t,t) \frac{d}{dt} \Phi'(t,t) \]  

(A-16)

Substituting eqs. (A-15) and (A-16) into eq. (A-12) we obtain

\[ Q(t) + K(t) B(t) R^{-1}(t) B'(t) K(t) = \]

\[ = K(t) \frac{d}{dt} \Phi(t,t) \cdot \Phi(t,t) - \Phi'(t,t) \frac{d}{dt} \Phi'(t,t) \cdot K(t) - \frac{d}{dt} K(t) \]  

(A-17)

Substituting eq. (17) into eq. (11) we deduce, using the property

\[ \Phi(t,t) \Phi'(t,t) = I, \]  

that

\[ J^* = \frac{1}{2} x' \Phi'(t_f,t) F \Phi(t_f,t) x - \frac{1}{2} x' \int_t^{t_f} \left[ \Phi'(t,t) K(t) \frac{d}{dt} \Phi(t,t) + \Phi'(t,t) K(t) \Phi(t,t) + \frac{d}{dt} K(t) \Phi(t,t) \right] dt \]  

x

(A-18)

But the integrand in (A-18) is simply equal to

\[ \frac{d}{dt} [\Phi'(t,t) K(t) \Phi(t,t)] \]  

(A-19)
as it can be readily verified by the chain rule. Hence,

\[ J^* = \frac{1}{2} x' \dot{\Phi}'(t_f, t) \Phi(t_f, t)x - \frac{1}{2} x' \left[ \Phi'(t_f, t) K(t_f) \Phi(t_f, t) - K(t) \right] x \]  

(A-20)

Since \( K(t_f) = F \), in view of Eq. (A-6), it follows that

\[ J^* = \frac{1}{2} x' K(t)x \]  

(A-21)

which is the desired result.  Q.E.D.
Model-Based Compensators for MIMO Control Synthesis and the LQG Algorithms

I. We now introduce a class of MIMO compensators which are called Model-Based Compensators (MBC) which have the property that when put together with any MIMO open-loop plant, the resultant closed-loop system will be stable, provided that we select the gain parameter appropriately. We will analyze the behavior of the closed-loop system when using MBC's. The mathematical analysis shows that MBC's can be constructed using results from optimal control theory (so-called linear-quadratic feedback problem) and from optimal estimation theory (the so-called Kalman filter). As a matter of fact, the so-called linear quadratic Gaussian (LQG) compensators are a subclass of MBC's.

The problems that a control system designer faces are important considerations in designing MIMO compensators. Given a possible unstable open-loop plant, the designer must select and implement a compensator that:

1. Attains and/or maintains stability of the closed-loop system.
2. Achieves desirable loop-shapes in the frequency domain using appropriate singular values.
Furthermore, the recently developed so-called \( H_{\infty} \)-optimal control can also be expressed as a subclass of MBC's. Particularly by selecting the controller parameters using \( H_{\infty} \)-optimal full state feedback and \( H_{\infty} \)-optimal state estimation, we can develop algorithms which select gains appropriate for MBC looking \( H_{\infty} \)-optimal compensators.
Up to this point we don't have a systematic procedure for achieving the stability objective given an arbitrary MIMO open-loop plant. We have the necessary tools for analyzing stability both in the time-domain (by matrix eigenvalues) and the frequency-domain (by the multivariable Nyquist criterion). However, we have not been able to extend, in an easy and natural manner, the SISO design concepts of root locus, Nyquist diagram etc., to the MIMO case in a systematic manner. That is we still do not know how to design a MIMO compensator \( K(s) \) for a feedback system, as follows:

![Feedback structure of a MIMO control system with disturbances \((d(s))\) reflected at the plant input.](image)

Given the nominal plant transfer matrix \( G(s) \), so that the resultant closed-loop system is stable, and we still have enough degrees of freedom for loop-shaping to achieve performance and maintain robustness.

Now we define a very special class of MIMO compensator, i.e., MBC's, in order to meet our MIMO objectives. These compensators contain the dynamics of the
Fig. 2. The Model Based Compensator in a Feedback Configuration.
nominal open-loop plant $G_p(s)$ in an explicit way. In addition, MBC's do contain several degrees of freedom, incorporated in two gain matrices, which can be adjusted to achieve the necessary performance-robustness tradeoffs while maintaining closed-loop stability.

Remark: The MBC structure has been derived from continuous progress in the area of robust optimal estimation and control theory. However, the word "optimal" should not be taken out of context because the results we will obtain, i.e., the compensator $K(s)$, will not be optimal in any sense.

II. The MBC:

Figure 2 shows an MBC internal structure along with an open-loop plant $G_p(s)$. The word MBC is because the matrices $A, B, C$ of the open-loop plant, appear also in the dynamics of the open compensator in a very specific way, almost duplicating the input-state-output interconnections in the plant.

The open-loop dynamics

We will assume that the open-loop dynamics can be described by the following vector differential equation:
\[ \dot{x}(t) = A x(t) + B u(t) + L d(t) \quad (\text{II-1}) \]
\[ y(t) = C x(t) \quad (\text{II-2}) \]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( d(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^n \).

The control-to-output transfer matrix is:
\[
G_p(s) = C (sI - A)^{-1} B \quad \text{so that,} \quad (\text{II-3})
\]
\[ y(s) = G_p(s) u(s) \quad (\text{II-4}) \]

and the disturbance to output relation is:
\[ y(s) = C (sI - A)^{-1} B \neq d(s) \quad (\text{II-5}) \]

so, that
\[
y(s) = C (sI - A)^{-1} B y(s) + C (sI - A)^{-1} \neq d(s) \quad (\text{II-5}^*)
\]

**The MBC Dynamics**

The vector \( z(t) \) is the state vector of the MBC and \( z(t) \in \mathbb{R}^n \), i.e., \( z(t) \) has the same dimension as the plant state \( x(t) \). Thus the MBC is an nth order system. The dynamics of the MBC can be written as:
\[
\dot{z}(t) = A z(t) + B u(t) + H y(t) \quad (\text{II-6})
\]
\[ y(t) = -g(t) - C z(t) = y(t) - C z(t) - z(t) \quad (\text{II-7}) \]
\[ u(t) = -C z(t) \quad (\text{II-8}) \]
Substituting (7) and (8) into (6) we have:

$$\dot{z}(t) = (A - BG - HC)z(t) - He(t) \quad (II-9)$$

The input to the MBC is the error $e(t)$ and the output of the MBC is the control $u(t)$. Thus (8) & (9) constitute an input-state-output description of the MBC. Therefore the MBC transfer function $K(s)$ defined by

$$y(s) = K(s) e(s) \quad (II-10)$$

is given by:

$$K(s) = G (sI - A + BG + HC)^{-1} H \quad (II-11)$$

Note that the two gain matrices $G$ and $H$ which represent the design parameters of the MBC, in effect change the poles and zeros of the $K(s)$.

### III. Closed-Loop System Dynamics

We can now calculate the closed-loop dynamics of the system either in the time or frequency domain. Apparently the time-domain analysis reveals with more insight.
A Closed-loop Representation

By substituting eq. \( \frac{d}{dt} \) into \( \frac{d}{dt} \) we obtain:

\[
\dot{x}(t) = Ax(t) - BCG\xi(t) + Ld(t) \\
y(t) = Cx(t)
\]  \( \text{(III-1)} \)

Since,

\[
\xi(t) = y(t) - \xi(t) = y(t) - Cx(t)
\]  \( \text{(III-2)} \)

Then, eq. (III-2) yields,

\[
\hat{z}(t) = HCGx(t) + [A - BCG - HCG]z(t) - HY(v(t)) \quad \text{(III-3)}
\]

Note eq. (III-1) and (III-3) represent a system of 2m + 2m differential equations; they can be written as:

\[
\begin{bmatrix}
\dot{X}(t) \\
\dot{Z}(t)
\end{bmatrix} =
\begin{bmatrix}
A & -BCG \\
HCG & A - BCG - HCG
\end{bmatrix}
\begin{bmatrix}
X(t) \\
Z(t)
\end{bmatrix} +
\begin{bmatrix}
L & 0 \\
0 & -H
\end{bmatrix}
\begin{bmatrix}
d(t) \\
v(t)
\end{bmatrix}
\]

\[\text{A}_{CL}\]  \( \text{(III-4)} \)

Eq. (III-5) describes the complete closed-loop system. Given \( A, B, C \), a particular selection of the gain matrices \( G \) and \( H \) will completely specify the closed-loop dynamics from which we can calculate the closed-loop response to disturbance and/or command inputs.
The stability of the closed-loop system will be assured if

$$\Re \{ \lambda_i[A_{cl}] \} < 0; \quad i = 1, \ldots, 2n.$$  (III-6)

However, the structure of the $2n \times 2n$ matrix $A_{cl}$ is not very transparent in (III-5), in order to deduce whether there exist matrices $C$ and $L$ that will imply equation (III-6).

A more transparent closed-loop representation

The features of the closed loop system of figure 2 can be made more transparent if we make a change of variables, which corresponds to a different state-space of the closed-loop dynamics.

The appropriate change of variables is:

$$W(t) = X(t) - Z(t).$$  (III-2)

It follows that:

$$W(t) = \dot{X}(t) - \dot{Z}(t) =$$

$$= (A - HC) W(t) + L \cdot d(t) + K \cdot W(t).$$  (III-8)

Therefore, the dynamics of the vector $W(t)$ are not coupled to the vector $X(t)$. Also (III-1) and (III-7) result in:

$$\dot{X}(t) = (A - BG) \cdot X(t) + BG \cdot W(t) + L \cdot d(t).$$  (III-9)
Thus, the dynamics of the closed-loop system are given by:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{w}(t)
\end{bmatrix} =
\begin{bmatrix}
A - BG & BG \\
0 & A - HC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix} +
\begin{bmatrix}
\frac{L}{2} & 0 \\
\frac{L}{2} & H
\end{bmatrix}
\begin{bmatrix}
d(t) \\
f(t)
\end{bmatrix}
\]

(III-10)

Using the determinant identity:

\[
\det \begin{bmatrix}
x & y \\
z & w
\end{bmatrix} = (\det x)(\det z),
\]

the 2n eigenvalues of the closed-loop system are the roots of the equation:

\[
0 = \det \begin{bmatrix}
\lambda I - A + BG & -BG \\
0 & \lambda I - A + HC
\end{bmatrix}
\]

\[
= \det (\lambda I - A + BG) \cdot \det (\lambda I - A + HC) \quad (III-11)
\]

Now, let

\[
\phi_1 (\lambda, G) = \det (\lambda I - A + BG) \quad (III-12)
\]
\[
\phi_2 (\lambda, H) = \det (\lambda I - A + HC) \quad (III-13)
\]

In order for the closed-loop system to be stable, we must have

\[
\text{Re } \lambda_i [A - BG] < 0, \quad i = 1, \ldots, n \quad (III-14)
\]

and

\[
\text{Re } \lambda_i [A - HC] < 0, \quad i = 2, 3, \ldots, 2n \quad (IV-15)
\]
Then the stability of the closed-loop system decomposes into two separate problems.

Problem 1. Given $A$ and $B$, can we find $x$ and $y$ such that

$$\text{Re} \left\{ \lambda_i (A - BC) \right\} < 0, \quad i = 1, \ldots, n?$$

Problem 2. Given $A$ and $C$, can we find an $H$ such that

$$\text{Re} \left\{ \lambda_i (A - HC) \right\} < 0, \quad i = 1, 2, \ldots, n?$$

It turns out that results available from linear system theory can be used to answer these problems in the affirmative. Let's now summarize these results.

II. Results from Linear System Theory

Definition II.1: The system

$$\dot{x}(t) = Ax(t) +Bu(t), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \quad (\text{II}-1)$$

is controllable, or $[A,B]$ is a controllable pair, iff

$$\text{rank} [B, AB, \ldots, A^{n-1}B] = n \quad (\text{II}-2)$$

The system (II-1) is stabilizable, or $[A,B]$ is a stabilizable pair, if the uncontrollable modes of the system are stable. Note that controllability implies stabilizability.
Theorem IV.4

Given the system
\[ \dot{x}(t) = A x(t) + B u(t) \] (IV-3)

Consider the full state variable feedback control law
\[ u(t) = -G x(t) \] (IV-4)

resulting in the closed-loop system
\[ \dot{x}(t) = [A - BG] x(t) = A_{cl} x(t) \] (IV-5)

(a) If \([A, B]\) is a controllable pair, then there exists at least one state feedback gain matrix \(G\) (there may be many) such that all of the closed-loop poles can be placed at arbitrary locations (subject to the complex conjugate) constraints. In particular, there exists at least one state feedback gain matrix \(G\) such that all closed-loop poles are at pre-specified locations in the left-half \(s\)-plane.

(b) If \([A, B]\) is only a stabilizable pair, then there exists at least one state feedback gain matrix \(G\) such that all closed-loop poles can be placed in the left-half \(s\)-plane.

Note: There are pole placement algorithms that give \([A, B]\) and desirable closed-loop poles, find the gain matrix \(G\). However, we will not discuss these.

Comment: Theorem IV.1 gives an affirmative answer to Problem 1 posed above.
In addressing Problem 2, we state the following definition.

**Definition IV. 2:** The system,

\[ \dot{x}(t) = A x(t) + u(t) \quad x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^p \quad (IV-6) \]

is observable, or the pair \([A, C]\) is an observable pair, if

\[ \text{rank} \left[ C, C A, \ldots, (C A)^{n-1} \right] = n \quad (IV-7) \]

The system \((IV-6)\) is detectable or \([A, C]\) is a detectable pair, if the unobservable modes of the system \((IV-6)\) are stable. Note that observability implies detectability.

**Theorem IV. 2:** Consider the system \((B = \mathbb{I})\)

\[
\dot{x}(t) = A x(t) + u(t) \quad x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^p \quad (IV-8)
\]

\[ y(t) = C x(t) \quad (IV-9) \]

(a) Suppose that \([A, C]\) is observable. Then there exists an output feedback gain matrix \(H\) (or maybe many)

\[ y(t) = -H y(t) = -H x(t) \quad (IV-10) \]

leading to the closed-loop system

\[ \dot{x}(t) = [A - H C] x(t) \quad (IV-11) \]
such that all the closed-loop eigenvalues \( \lambda_i [A - HC] \) can be placed in prespecified locations.

(b) Suppose that \([A,C]\) is detectable. Then there exists at least one output gain matrix \(H\) such that all closed-loop poles \( \lambda_i [A - HC] \) of (4.4.4) are in the left-half \(s\)-plane.

Note: This theorem can be proved by using the duality of controllability and observability.

Comment: Theorem (IV.2) basically answers Problem 2 in an affirmative fashion. This theorem is also central in the derivation of a stable

\[ \text{Fig. VI.1 Visualization of the stable control system implied by Theorem VI.1 if } [A,B] \text{ is stabilizable under full state variable feedback.} \]
Fig. IV.2 Visualization of the stable control system implied by Theorem IV.2 if \([A, C]\) is detectable under output feedback.

Fig. IV.3 Definition of the observer problem. If \([A, C]\) is detectable and \(\text{Re } \lambda_i [A-\mu C] < 0\) then,
\[
\lim_{t \to \infty} \hat{x}(t) = x(t), \quad \text{and } \lim_{t \to \infty} \hat{y}(y - y(t))
\]

Note: The observer problem consists of deducing a dynamic system (observer) driven by the output \(y(t)\) of the plant with the property that the observer state vector \(\hat{x}(t)\) approaches the plant state \(x(t)\) as \(t \to \infty\). As shown the observer involves a feedback loop of the form shown in Fig IV.2. The observer gain matrix \(K\) is selected such that the observer dynamics are
stable, i.e. \( \Re \, \Re^* \{ A - iC \} < 0 \).

II. A "Cook Book" Calculation of the \( G \) and \( H \) Gain Matrices (LQG compensator)

We have shown that the feedback loop of Fig. 2 can be made closed-loop stable by the appropriate selection of the two gain matrices \( G \) and \( H \).

Now we present a procedure for actually evaluating numerical values for \( G \) and \( H \) using available computer tools. This procedure is based on the LQG stochastic optimal control theory. Therefore, the LQG-based compensators are a special class of MBC's.

VI. Calculating the \( G \) Gain matrix (control gain) using the Linear-Quadratic (LQ) algorithm.

Let's start with the \( n \times n \) matrix \( A \) of \( n \times n \) matrix \( B \); we assume that \( [A, B] \) is stabilizable.

**Step 1:** Select an arbitrary \( n \times n \) symmetric and positive definite matrix denoted by \( R \), i.e.

\[
R = R^T > 0
\]  \hspace{1cm} (VI.1)

**Step 2:** Select an arbitrary \( p \times n \) matrix denoted by \( N \) with the only property that \( [A, N] \) is detectable
Let $\mathbf{Q}$ denote the $m \times m$ matrix

$$\mathbf{Q} = \mathbf{N}' \mathbf{N}$$

which has the property

$$\mathbf{Q} = \mathbf{Q}' > 0$$

Remark: $\mathbf{N}$ is called a square root matrix of $\mathbf{Q}$.

**Step 3:** Let the computer subroutines solve for the unique positive semi-definite symmetric $m \times m$ solution matrix $\mathbf{K}$, of the so-called control algebraic Riccati equation (CARE).

$$- \mathbf{K} \mathbf{A} - \mathbf{A}' \mathbf{K} - \mathbf{Q} + \mathbf{K} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}' \mathbf{K} = 0$$

(People usually pick $\mathbf{Q} = \mathbf{C} \mathbf{C}'$ and $\mathbf{R} = \mathbf{I}$)

**Step 4:** Let also the computer subroutines calculate the $mxn$ gain matrix $\mathbf{G}$ by the formula

$$\mathbf{G} = \mathbf{R}^{-1} \mathbf{B} \mathbf{K}$$

Remark: The assumption of the above steps guarantee that

$$\text{Re} \, \mathbf{x}_i \left[ \mathbf{A} - \mathbf{B} \mathbf{G} \right] < 0 ; \quad i = 1, 2, \ldots, m$$

no matter what are the numerical values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$ and $\mathbf{K}$.

**V.2** Calculating the $H_2$ gain matrix (filter gain) using the Kalman filter algorithm (KF).

We start with the $m \times n$ matrix $\mathbf{A}$ and the $m \times n$
matrix $\Xi$; we assume that $[\Phi \Xi]$ is detectable.

**Step 1:** Select an arbitrary $m \times m$ symmetric and positive definite matrix denoted by $\Theta$; i.e.,
\[ \Theta = \Theta' > 0 \]  
(V.8)

**Step 2:** Select an arbitrary $m \times m$ matrix $M$ with the only property that $[\Phi M]$ is stabilizable. Let $\Xi$ denote the $m \times m$ matrix
\[ \Xi = M M' \]  
(V.8)

which has the property
\[ \Xi = \Xi' > 0 \]  
(V.9)

Note that $M'$ is the square root of $\Xi$.

**Step 3:** Let the computer subroutines solve for the unique positive-semidefinite symmetric $m \times m$ solution matrix $\Xi$ of the so-called filter algebraic Riccati equation (FARE).

People usually pick $\Xi = LL'$ and $\Theta = I$

\[ A \Xi + \Xi A' + \Xi - \Xi C' \Theta^{-1} C \Xi = 0 \]  
(V.10)

**Step 4:** The computer subroutines will also calculate the $m \times m$ gain matrix $M$ by:
\[ M = \Xi C' \Theta^{-1} \]  
(V.11)

Remarks: The assumptions and the above steps guarantee that:
\[ \text{Re} \, \theta_i [A - HC] < 0, \quad i = 1, 2, \ldots, n \]  
(V.12)
no matter what are the numerical values of \( n, m, A, C, \in D \).

**Summary:** When the gain matrices \( G \) and \( H \) of the MBC are calculated by the above procedure, then we refer to the MBC as a Linear Quadratic Gaussian (LQG) compensator. We stress that an LQG compensator guarantees the closed-loop stability of the feedback control system because the properties (V.6) and (V.12) are guaranteed and \( \lambda_i [A - BG] \) and \( \lambda_i [A - HC] \) define the 2n closed-loop poles.

**VII. The Closed-Loop Transfer Matrices**

**Input to output:**
\[
\mathbf{y}(s) = \mathbf{G}_{cl}(s) \mathbf{r}(s) \tag{VII.1}
\]
\[
\mathbf{y}(s) = \mathbf{C} (sI - A + BG)^{-1} BG (sI - A + HC)^{-1} H \mathbf{r}(s) \tag{VII.2}
\]
\[
\mathbf{G}_{cl}(s)
\]

**Disturbance to output:**
\[
\mathbf{y}(s) = \mathbf{G}_0 (s) \mathbf{d}(s) \tag{VII.3}
\]
\[
\mathbf{y}(s) = \mathbf{C} (sI - A + BG)^{-1} [I - BG (sI - A + HC)^{-1}] \mathbf{d}(s) \tag{VII.4}
\]
\[
\mathbf{G}_0 (s)
\]
VI. A "Cook Book" Calculation of the $G$ and $H$ Gain Matrices (Hoo compensator)

The previous section demonstrated the use of LQG (or time optimal stochastic control theory) for finding the gains $G$ and $H$ of the HBC.

An implicit assumption in the LQG development is the nature of the disturbance $d(t)$, which is assumed to be white Gaussian stochastic process. However, in numerous applications, the external disturbances are not well-approximated by white Gaussian noise. As a matter of fact in numerous applications the disturbances are closely approximated by with deterministic models.

Now we present a procedure for actually calculating the numerical values of $G$ and $H$ using available computer tools. This procedure is based on the Hoo-optimal control theory. Therefore, Hoo-optimal compensators are a special class of HBC's.
II.1 Calculating the G gain matrix (Control gain) using the $H_\infty$ optimal algorithm.

Refresh memory:
System dynamics:
$$\dot{x}(t) = Ax(t) + B u(t) + L d(t)$$
$$y(t) = C x(t)$$

Let's start with the $m \times n$ matrix $A$ and $n \times k$ matrix $B$.
We assume that $[A, B]$ is stabilizable.

Step 1: Select an arbitrary $m \times m$ symmetric and positive definite matrix denoted by $R$, i.e.
$$R = R' > 0$$

Step 2: Select an arbitrary $p \times n$ ($p < n$) matrix denoted by $N$ with the only property that $[A, N]$ is detectable. Let $Q$ denote the $m \times n$ matrix:
$$Q : N' \cdot N$$

Which has the property
$$Q = Q' > 0$$

Remark: $N$ is called a square root of matrix $Q$. 
Step 3: Let the computer subroutines solve for the unique positive semi-definite symmetric \( m \times m \) solution matrix \( K \), of the so-called modified control algebraic Riccati eq. (MARE)

\[-KA - A'K - Q + K(BR^{-1}B' - L'L)^TK = 0 \quad (V1.4)\]

(usually \( Q = C'C \) and \( R = I \))

Step 4: Let also the computer subroutines calculate the \( m \times n \) gain matrix \( G \) by the formula:

\[
G = R^{-1/2}K
\quad (V1.5)
\]

Remark: The assumptions of the above step guarantee that:

\[
A + \left( \frac{1}{2}LL' - BR'B' \right) K = A - BG + \frac{1}{2}L'L'K
\]

\[
\operatorname{Re} Z_i [A - BG] < 0 \quad ; \quad i = 1, \ldots, n \quad (V1.6)
\]

no matter what are the numerical values of \( n, m \), \( A, B, L, Q \leq R \) and \( \gamma (\geq 0) \)

\[\text{III.2} \quad \text{Calculating the H gain matrix (filter gain) using the observer algorithm.} \]

Start with \( n \times n \) matrix \( A \) and the \( m \times n \) matrix \( C \); we assume that \([A, C]\) is detectable.
Step 1: Select an arbitrary \( n \times n \) symmetric and positive definite matrix denoted by \( \Theta \); i.e.,
\[
\Theta = \Theta' > 0
\]  
\[\text{(VI.7)}\]

Step 2: Select an arbitrary \( n \times p \) matrix \( M \) with the only property that \([-A, M]\) is stabilizable. Let \( \Xi \) be the \( n \times n \) matrix
\[
\Xi = MM'
\]  
\[\text{(VI.8)}\]

which has the property
\[
\Xi = \Xi' > 0
\]  
\[\text{(VI.9)}\]

Note \( M' \) is the square root of \( \Xi \).

Step 3: Let the computer routines solve for the unique positive semidefinite symmetric \( n \times n \) solution matrix \( \Sigma \), of the so-called modified filter algebraic Riccati equation (FARE)
\[
A\Sigma + \Sigma A' + \Xi + \Sigma \left( \frac{1}{2} C'C - C' \Theta^{-1} C \right) \Sigma = 0
\]  
\[\text{(VI.10)}\]

(Usually people pick \( \Xi = L^{-1} L' \) and \( \Theta = \Theta' = I \)).

Step 4: The gain \( H \) is then given by:
\[
H = \Sigma C' \Theta^{-1}
\]  
\[\text{(VI.11)}\]

Remark: The above assumptions and the above steps guarantee that
\[
\text{Re} \Theta \left[ A + \Sigma \left( \frac{1}{2} C'C - C' \Theta^{-1} C \right) \right] = \text{Re} \Theta \left[ A - HC + \Xi \frac{1}{2} C'C \right] < 0
\]
no matter what are the values of \( n, m, A, c, \in, \Theta \) and \( f \).

**Summary:** When the gain matrices \( G \) and \( H \) of the HBC are calculated by the above procedure, then we refer to the HBC as an \( H_{\infty} \)-compensator. We stress that an \( H_{\infty} \)-compensator guarantees that:

\[
\| T_{dy} \|_{\infty} \leq f
\]

i.e., max amplification is less than \( f \), in addition to closed-loop stability.
III. Can we stop here?

Since we now have two independent ways of calculating the matrices $G$ and $H$ we can design a MIMO compensator that has the CBC structure.

In particular we can design either LQG or MIMO compensators which will be nominally stable (always) and by "tuning" the $G$ and $H$ make them have certain desired performance characteristics. However, it was determined that any pair of $G$ and $H$ that results from the above two algorithms does not lead to desirable compensators. In particular, it was determined that arbitrary LQG (and as a matter of fact the) compensators exhibit very poor robustness. That is if the actual system has dynamics not modeled in the design process, implementation of the resulting compensator quite often leads to undesirable closed-loop feedback loops.

Therefore, people have started looking at ways to restrict the selection of $G$ and $H$ to those matrices which, in addition to closed-loop stability of performance, also guarantee some degree of stability robustness (or even performance robustness).
JANUARY 10, 1991:

- INTRODUCTION & OBJECTIVES
- HISTORICAL OVERVIEW
- MATHEMATICAL BACKGROUND
- UNCERTAINTY MODELING
- ISSUES IN MULTIVARIABLE CONTROL SYSTEMS
  - LQR, KF, LQG
  - LQ-Servo, LQG/LTR
ADAPTATION OF THE LQR DESIGNS TO COMMAND-FOLLOWING AND OUTPUT-DISTURBANCE REJECTION

FEEDBACK CONTROL SYSTEMS

THE STANDARD LQR LOOP

\[
\begin{align*}
\mathbf{x}(t) &= A \mathbf{x}(t) + B \mathbf{u}(t) \\
\mathbf{x}(t) &\in \mathbb{R}^n \quad \mathbf{u}(t) \in \mathbb{R}^m
\end{align*}
\]

\[
\mathbf{u}(t) = -G \mathbf{x}(t)
\]

If we break the loop at \( \mathcal{O} \), i.e. plant input, then, the LTI say \( T_f(s) \) is the LQR TFM, i.e.

\[
T_f(s) = G_{LQ}(s) = G (sI - A)^{-1} B
\]

Under mild assumptions,

(1) the primal system is nominally stable,

(2) the TFM \( G_{LQ}(s) \) has nice crossover properties as exhibited by the s.v. inequalities:

\[
\begin{align*}
\delta_{\min} (I + T_f(j\omega)) &> 1 \\
\delta_{\min} (I + T_f^{-1}(j\omega)) &> \frac{1}{2}
\end{align*}
\]
which imply good robustness properties when the
modeling errors are reflected at the plant
input, i.e., point 0.

(2) we can shape the \( S(V) \) of \( \mathbf{u} \) by
cleverly selecting the design parameters \( Q \) and
\( R \) of the LQR problem.

In spite of all these nice things it is not
desirable how to adapt the LQR for:

(1) command-following
(2) output disturbance rejection.

These are precisely the performance-oriented require-
ments of MIMO servomechanisms.

The problem with this loop is that the error
signal is outside the feedback loop and therefore
we can not accomplish good command following and
output-disturbance rejection.
It is possible to address the CF of the Reg. problem using a variant of the LQ optimal control problem; however, its solution has some unpleasant properties. These properties are as follows:

The optimal control $u(t)$, at the present time $t$, depends on:

(a) the present value of state $x(t), \quad 0 \leq t \leq K$

(b) the future values of $r(t), \quad t \leq t < \infty$ and

(c) the future values of $d(t), \quad t \leq t < \infty$.

In most practical situations, we don’t know nor can we measure $r(t)$ or $d(t)$. The bottom line is that we can’t use the solution of a dynamic optimal control problem unless we know the future reference & disturbances.

**Setting up the LQ servo problem**

Our plant is an LTI MIMO system with control $u(t)$ & output $y(t), \quad u \in \mathbb{R}^m, y \in \mathbb{R}^n$. Let $r(t) \in \mathbb{R}^m$ be the reference command input & $d(t) \in \mathbb{R}^n$ is the output disturbance. Then, the output to be controlled is

$$y(t) = y_p(t) + d(t)$$

and the error vector is

$$e(t) = r(t) - y(t)$$
We assume that the vector $y_p$ is directly measured, therefore, $y_p(t)$ is a subvector of $x(t)$.

So,

$$
x(t) = \begin{bmatrix}
  y_p(t) \\
  x_r(t)
\end{bmatrix}; \quad x \in \mathbb{R}^n, \ y_p \in \mathbb{R}^m, \ y_r \in \mathbb{R}^{n-m}
$$

The plant model is given by,

$$
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y_p(t) &= \begin{bmatrix} I_m & 0 \end{bmatrix} x(t)
\end{align*}
$$

Suppose how we carry an LQR design which results to:

$$
y(t) = -G x(t)
$$

Now according to the decomposition of the state vector $x$, the following can be said about $G$:

$$
G = \begin{bmatrix} G_y & G_r \\ m \times m & m(n-m) \end{bmatrix}
$$

or,

$$
y(t) = -G_y y_p - G_r x_r(t)
$$

and we can now redraw the standard LQR loop as:

Note: Both $y_p(t)$ and $x_r(t)$ are assumed measured.
Another way of visualizing the LQG loop, which is identical to the previous one. Indeed, the TF at 1 is $T_1(s) = \frac{Gy}{s}$. It should be obvious why this is done. At the location of 2 we will have $r \neq d$. Thus, we can define an LQ-servo loop as follows:

Notice that the error signal appears naturally in the feedback loop.

Remarks:
1. The LQ-servo loop is not optimal with respect to anything related to $e(t)$ or $d(t)$.
2. The nominal LQ-servo loop is guaranteed to be stable. (Closed-loop poles $\lambda_i \in \Re - \mathbb{C}_+$.)
3. The LQ regulator can be used to see how the LQ-servo works. The LQK sees any non-zero $y_0$ or $x_0$ as "bad" and adjusts the control to drive them to zero. So, any signal that appears in front of $G_y$ is interpreted as non-zero $y_0$. 
Now, the servo, in the presence of \( \epsilon \) and \( \delta \) has a nonzero \( \epsilon(t) \) and the signal entering \( G_y \) is forced to become zero. The servo thinks that \( \epsilon(t) \) is up and tries to set it to zero, but this accomplishes our goal because this is what we want. Thus we have used the solution of the \( \epsilon \) problem and we obtained a suboptimal servo that works!

**Stability- Robustness Issues**

If we reflect high-frequency modeling uncertainty to the plant input then the LTI of interest is \( T_1(s) \), i.e.

\[
T_1(s) = G_{\text{eq}}(s) = G(15\mathbf{I} - A)^{-1}\mathbf{B}
\]

We have several tricks by which we can shape the SV's of \( T_1(s) \). However, if we focus our attention only on the SV's of \( T_1(s) \) then:

(a) we obtain a realistic picture of the stability robustness of the system and in particular to high-frequency modeling errors.

(b) we can obtain an erroneous idea of the command-following and output disturbance rejection properties of the servo.
Performance Issues

Whether or not the LC servo does a good job in disturbance rejection in the low-frequency range will depend on the SV's of $T_2(s)$, i.e., if we break the loop at $\mathcal{G}$. It's easy to see that even though $T_1$ and $T_2$ are $m \times m$ matrices, they are generally different. $T_2(s)$ can be calculated to be:

$$T_2(s) = G_p (sI - A + B G_r D_p)^{-1} B G_r$$

Let's define, $Q(s)$, the TF from $y_r(t)$ to $y_p(t)$, i.e.,

$$Q(s) = G_p (sI - A + B G_r D_p)^{-1} B G_r$$

Now, the plant is given by:

$$\dot{x} = A x + B u$$

and

$$y_p = G_p x$$

So,

$$\dot{y}_p = y_r(t) - \dot{y}_r(t) = y_e(t) - G_r \dot{x}_r(t)$$

If we define $D_p = \begin{bmatrix} 0 \\ \Pi \end{bmatrix}$, then,

$$\dot{x}_r = D_p x$$

or

$$\dot{x} = A x - B G_r D_p x + B y_e = (A - B G_r D_p) x + B y_e$$

So,

$$Q(s) = G_p (sI - A + B G_r D_p)^{-1} B$$
where,

\[ G_p = \begin{bmatrix} I_{n \times n} & 0_{m \times (m-n)} \end{bmatrix} \]

Therefore the CF & OR properties of the LD-zero
are characterized by the shapes of the SV's of
\[ I_2(s) \].

Remarks:
(1) If we shape the SV's of \[ I_2(s) \], then we
have effectively calculated \[ G_y & G_r \] in the LD-zero
problem. Hence, the SV's of \[ I_2 \] are fixed; we have
no other degree of freedom. Once we design a
good \[ I_2(s) \] we are stuck with the resulting \[ I_2(s) \].

(2) As we know we can use the FD equality for
LD regulators to shape \( 0_{i}(I_2(s)) \). However,
there is no easy way of shaping the SV's of
\( 0_{i}(I_2(s)) \) even only in the low frequency region.

(3) A limited amount of experience indicates that
if \( G_p(sI-A)^{-1}B \) is minimum phase then, if one
makes \( 0_{min}[I_2(s)] \) large at low freq. then \( 0_{min}[I_2(s)] \)
will also be large in the same frequency range.
But not the same. However for Non-min. phase
plants this is not the case.

Conclusion: The designer must shape \( I_2(s) \) and
see what happens to \( I_2(s) \).
LECTURE NOTES
ON
THE KALMAN FILTER

April 5, 1983

Written by
MICHAEL ATHANS

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THE KALMAN FILTER

0. SUMMARY

The purpose of this chapter is to discuss the celebrated Kalman filter (KF). The Kalman filter is of fundamental importance on its own right in the field of stochastic dynamic optimal estimation. However, our discussion of the Kalman filter will be slanted towards the way that we are going to use it in MIMO control system design; it is a dynamic MIMO system that will define the so-called Linear-Quadratic-Gaussian (LQG) model-based compensator.

In fact we want to stress at the outset that KF "purists" will be shocked by the way that we plan to adjust the KF free parameters, with total disregard about the mathematical definitions and paying absolutely no attention to stochastic optimality considerations. Thus, we shall use the KF as a means to an end; in particular, how it will help us

(a) shape loop singular values, and
(b) design dynamic MIMO compensators for feedback control.

1. PROBLEM DEFINITION

In this section we present the definition of the basic stochastic estimation problem for which the KF yields the "optimal" answer.

1.1 State Dynamics

Consider an LTI stochastic dynamic system whose state vector $x(t)$ obeys the stochastic differential equation

$$\dot{x}(t) = A x(t) + B u(t) + L \xi(t)$$  \hspace{1cm} (1-1)
In Eq. (1-1) \( u(t) \) is a deterministic (control) input, assumed known. The vector \( \xi(t) \) is a stochastic process called the process (or plant) noise, with certain key properties. In particular, we assume that \( \xi(t) \) is a continuous-time white Gaussian noise. Its mathematical characterization is as follows: it has zero-mean for all \( t \), i.e.

\[
E\{\xi(t)\} = 0
\]

and its covariance matrix at times \( t \) and \( \tau \) is defined by

\[
\text{cov}\{\xi(t);\xi(\tau)\} \triangleq E\{\xi(t)\xi(\tau)^T\} = \Xi \delta(t-\tau)
\]

(1-3)

with \( \delta(t-\tau) \) being the Dirac delta function (impulse at \( t=\tau \)). The matrix \( \Xi \) is called the intensity matrix of \( \xi(t) \) and it has the following properties

\[
\Xi = \Xi^T > 0
\]

(1-4)

Remark: Continuous white noise does not exist in nature; it is the limit of a broad-band noise process. Continuous white noise has constant power at all frequencies; hence, infinite energy (this is why it does not really exist). It is completely unpredictable since from Eq. (1-3) it follows that is uncorrelated for any \( t \neq \tau \), while it has infinite variance at \( t=\tau \). Nonetheless, it is an excellent modeling tool.

Remark: The state \( \chi(t) \), the solution of Eq. (1-1) is a well-defined physical process; it actually is a colored Gaussian random process.

1.2 Measurement Equation

We assume that our sensors cannot directly measure all of the physical state variables, the components of \( \chi(t) \), of the system (1-1). Rather we assume that our sensors can only measure certain output variables in the presence of additive measurement noise.
The mathematical model of the measurements is as follows:

\[ y(t) = C \ x(t) + \theta(t) \]  (1-5)

The vector \( y(t) \) is the actual sensor measurement. The measurement (or sensor) noise \( \theta(t) \) is assumed to be a continuous-time white Gaussian noise process, independent of \( \xi(t) \), with zero mean, i.e.

\[ E(\theta(t)) = 0 \]  (1-6)

and covariance matrix

\[ \text{cov}[\theta(t); \theta'(t)] = E(\theta(t)\theta'(t)) = \Theta \delta(t-t) \]  (1-7)

with

\[ \Theta = \Theta' > 0 \]  (1-8)

Figure 1-1 shows a visualization, in block diagram form, of Eqs. (1-1) and (1-5).

1.3 The State Estimation Problem

Let imagine that we have been observing the control \( u(t) \) and the output \( y(t) \) over the infinite past up to the present time \( t \). Let

\[ U(t) = \{u(t); -\infty < t \leq t\} \]  (1-9)

\[ Y(t) = \{y(t); -\infty < t \leq t\} \]  (1-10)

denote the past histories of the control and output, respectively.

The state estimation problem is as follows: Given \( U(t) \) and \( Y(t) \) find a vector \( \hat{x}(t) \), at time \( t \), which is an "optimal" estimate of the present state \( x(t) \) of the system (1-1).
A stochastic linear dynamic system

Fig. 1
Under the stated assumptions regarding the Gaussian nature of \( \xi(t) \) and \( \eta(t) \), the "optimal" state estimate is the conditional mean of the state, i.e.

\[
\hat{x}(t) = E[x(t) \mid U(t), Y(t)]
\]  
(1-11)

One can relax the Gaussian assumption and define the optimality of the state estimate \( \hat{x}(t) \) in different ways.

One popular way is to demand that \( \hat{x}(t) \) be generated by a linear transformation on the past "data" \( U(t) \) and \( Y(t) \), such that the state estimation error \( \bar{x}(t) \)

\[
\bar{x}(t) \overset{\Delta}{=} x(t) - \hat{x}(t)
\]  
(1-12)

has zero mean, i.e.

\[
E[\bar{x}(t)] = 0
\]  
(1-13)

and the cost functional

\[
J = E \left\{ \sum_{i=1}^{n} \bar{x}_i^2(t) \right\} = E[\bar{x}(t)\bar{x}(t)'] = \text{tr}[E[\bar{x}(t)\bar{x}'(t)]]
\]  
(1-14)

is minimized.

The cost functional \( J \) has the physical interpretation that it minimizes the sum of the error variances \( E[\bar{x}_i^2(t)] \) for each state variable. If we let \( \Sigma \) denote the covariance matrix (stationary) of the state estimation error

\[
\Sigma \overset{\Delta}{=} E[\bar{x}(t)\bar{x}'(t)]
\]  
(1-15)

then the cost \( J \) of Eq. (1-14) can also be written as

\[
J = \text{tr}[\Sigma]
\]  
(1-16)
**Bottom Line:** We need an algorithm that translates the signals we can observe, \( u(t) \) and \( y(t) \), into a state estimate \( \hat{x}(t) \), such that the state estimation error \( \dot{x}(t) \) is "small" in some well-defined sense. The KF is the algorithm that does just that!

2. **SUMMARY OF THE KALMAN FILTER EQUATIONS**

2.1 **Additional Assumptions**

In this section we summarize the on-line and off-line equations that define the Kalman filter. Before we do that we make two additional "mild" assumptions

\[
[A, L] \text{ is stabilizable (or controllable)} \quad (2-1)
\]

\[
[A, C] \text{ is detectable (or observable)} \quad (2-2)
\]

If \([A, L]\) is controllable, this means that the process noise \( \tilde{x}(t) \) excites all modes of the system (1-1); if \([A, C]\) is observable that means that the "noiseless" output \( y(t) = C \, x(t) \) contains information about all state variables.

2.2 **The Kalman Filter Dynamics**

The function of the KF is to generate in real-time the state estimate \( \hat{x}(t) \) of the state \( x(t) \). It actually is an LTI dynamic system, of identical order \( n \) to the plant (1-1), and is driven by

(a) the deterministic control input \( u(t) \), and

(b) the measured output vector \( y(t) \).

The Kalman filter dynamics are given as follows:

\[
\dot{\hat{x}}(t) = A \, \hat{x}(t) + B \, u(t) + \, H[y(t) - C \, \hat{x}(t)]
\]  
\[(2-3)\]
A block diagram visualization of Eq. (2-3) is shown in Fig. 2-1. Note that in
Eq. (2-3) all variables have been defined previously except for the KF gain matrix
H whose calculation is carried out off-line and will be discussed in Section 2.4.

The filter gain matrix H multiplies the so-called residual or innovations
vector

\[ r(t) \triangleq y(t) - C \hat{x}(t) \]  \hspace{1cm} (2-4)

and updates the time-rate-of-change, \( \dot{\hat{x}}(t) \), of the state estimate \( \hat{x}(t) \). The
residual \( r(t) \) is like an "error" between the measured output \( y(t) \), and the
predicted output \( C \hat{x}(t) \).

Remark: From an intuitive point of view the KF, defined by Eq. (2-3) and
illustrated in Fig. 2-1, can be thought as a model-based observer or state-
reconstructor. The reader should carefully compare the structures depicted
in Figs. 1-1 and 2-1. The plant/sensor properties, reflected by the matrices
A, B and C, are duplicated in the KF*. Indeed the optimal predictor is obtained
with \( H=0 \). The state estimate \( \hat{x}(t) \) is continuously updated by the actual sensor
measurements, through the formation of the residual \( r(t) \) and "closing" the loop
with the filter gain matrix H.

The KF dynamics of Eq. (2-3) can also be written in the form

\[ \dot{\hat{x}}(t) = [A - H C] \hat{x}(t) + B u(t) + H y(t) \]  \hspace{1cm} (2-5)

* No signal corresponding to \( L \xi(t) \) shows up in Fig. 2-1. This is because we
assumed that \( \xi(t) \) had zero-mean; being completely unpredictable its, "best
guess" is 0.
Fig 2-1. The structure of the Kalman Filter. The control, $u(t)$, and measured output, $y(t)$, are those associated with the stochastic system of Fig 1-1. The filter gain matrix $H$ is computed in a special way.
From the structure of Eq. (2-5) we can immediately see that the stability of the KF is governed by the matrix \([A-H \ C]\). At this point of our development we remark that Assumption (2-2), i.e. the detectability of \([A, C]\), guarantees the existence of at least one filter gain matrix \(H\) such that the KF is stable, i.e.

\[
\text{Re } \lambda_1 [A-H \ C] < 0; \quad i=1,2,\ldots,n \tag{2-6}
\]

2.3 Properties of Model-Based Observers

We have remarked that the KF gain matrix \(H\) is calculated in a very special way. However, it is extremely useful, and consistent with our Model-Based-Compensator (MBC) philosophy, to examine the structure of Eqs. (2-5) or (2-5) and Fig. 2-1 with an arbitrary filter gain matrix \(H\), except that it leads to a stable loop in the sense that Eq. (2-6) holds. Thus, for the development that follows in this subsection think of \(H\) as being a fixed matrix.

As before let \(\hat{x}(t)\) denote the state estimation error vector

\[
\hat{x}(t) \triangleq x(t) - \hat{x}(t) \tag{2-7}
\]

It follows that

\[
\dot{\hat{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \tag{2-8}
\]

Next, we substitute Eqs. (1-1), (1-5) and (2-5) into Eq. (2-8) and use Eq. (2-7) as appropriate. After some easy algebraic manipulations we obtain the following stochastic vector differential equation for the state estimation error \(\hat{x}(t)\):

\[
\dot{\hat{x}}(t) = [A-H \ C]\hat{x}(t) + L \xi(t) - H \theta(t) \tag{2-9}
\]

Note that, in view of Eq. (2-6), the estimation error dynamic system is stable. Also note that the deterministic signal \(B \ u(t)\) does not appear in the error equation (2-9).
Under our assumptions that the system is stable and it was started at the indefinite past \((t_0 \rightarrow -\infty)\), then it is easy to verify that

\[
E(\bar{x}(t)) = 0
\]  

(2-10)

This implies that any stable model-based estimator of the form shown in Fig. 2-1, with any filter gain matrix \(H\), gives us unbiased estimates.

Using next elementary facts from stochastic linear system theory one can calculate the error covariance matrix \(\Sigma\) of the state estimation error \(\bar{x}(t)\)

\[
\Sigma = \text{cov}[\bar{x}(t); \bar{x}(t)] = E[\bar{x}(t)\bar{x}'(t)]
\]  

(2-11)

It is the solution of the so-called Lyapunov matrix equation (linear in \(\Sigma\))

\[
[A - HH'] \Sigma + \Sigma [A - HH']' + L \bar{E} L' + H \odot H' = 0
\]  

(2-12)

with

\[
\Sigma = \Sigma' > 0
\]  

(2-13)

Thus, for any given filter gain matrix \(H\) we can calculate\(^*\) the associated error covariance matrix \(\Sigma\) from Eq. (2-12). Recalling the discussion of Section 1.3, we can evaluate, for a given \(H\), the quality of the estimator by calculating

\[
J = \text{tr}[\Sigma]
\]  

(2-14)

The specific way that the KF gain is calculated is by solving a constrained static optimization problem of minimizing (2-14) with respect to the elements \(h_{ij}\) of the matrix \(H\) subject to the algebraic constraints (2-12) and (2-13).

\(^*\) The HONEY-X software package can solve Lyapunov equations.
2.4 The Kalman Filter Gain and Associated Filter Algebraic Riccati Equation (FARE)

We now summarize the off-line calculations* that define fully the Kalman filter (2-3) or (2-5).

The KF gain matrix $H$ is computed by

$$
H = \Sigma C' \Theta^{-1},
$$

(2-15)

where $\Sigma$ is the unique, symmetric, and at least positive semidefinite solution matrix of the so-called Filter Algebraic Riccati Equation (FARE)

$$
0 = A \Sigma + \Sigma A' + L \Sigma L' - \Sigma C' \Theta^{-1} C \Sigma
$$

(2-16)

with

$$
\Sigma = \Sigma' > 0
$$

(2-17)

Remark: The KF gain is obtained by setting

$$
\frac{\partial}{\partial h_{ij}} \text{tr}(\Sigma) = 0
$$

(2-18)

where $\Sigma$ is given by Eq. (2-12). The result is Eq. (2-15). Substituting Eq. (2-15) into Eq. (2-12) one deduces the FARE (2-16).

2.5 Duality Between the KF and LQ Problems

The mathematical problems associated with the solution of the LQ and KF are dual. This duality was recognized by R.E. Kalman as early as 1960.

The duality can be used to deduce several properties of the KF simply by "dualizing" the results of the LQ problem. See Table 2.1. A summary of the KF properties is given by Section 3.

*These calculations can be executed by HONEY-X.
<table>
<thead>
<tr>
<th>LQ Problem</th>
<th>KF Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$-A'$</td>
</tr>
<tr>
<td>$B$</td>
<td>$C'$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$L \equiv L'$</td>
</tr>
<tr>
<td>$R$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$K$</td>
<td>$\Sigma$</td>
</tr>
<tr>
<td>$[A, Q^{1/2}]$ detectable</td>
<td>$[A, L]$ stabilizable (with $\Sigma = I$)</td>
</tr>
</tbody>
</table>
3. KALMAN FILTER PROPERTIES

3.1 Introduction

In this section we summarize the key properties of the Kalman filter. These properties are the "dual" of those derived for the LQ controller.

3.2 Guaranteed Stability

Recall that the KF algorithm is

\[ \dot{x}(t) = [A-H \ C] \hat{x}(t) + B \ u(t) + H \ y(t) \]  \hspace{1cm} (3-1)

Then, under the assumptions of Section 2, the matrix \([A-H \ C]\) is strictly stable, i.e.

\[ \text{Re} \ \lambda_i [A-H \ C] < 0; \ i=1,2,...,n \]  \hspace{1cm} (3-2)

3.3 Frequency Domain Equality

One can readily derive a frequency domain* equality for the KF. In the development that follows let

\[ \Xi = I \]  \hspace{1cm} (3-3)

Let us make the following definitions: Let \( G_{KF}(s) \) denote the KF loop-transfer matrix

\[ G_{KF}(s) \overset{\Delta}{=} C (sI-A)^{-1} H \]  \hspace{1cm} (3-4)

\[ G_{KF}^H(s) \overset{\Delta}{=} H' (-sI-A')^{-1} C' \]  \hspace{1cm} (3-5)

Let \( G_{FOL}(s) \) denote the filter open-loop transfer matrix (from \( \xi(t) \) to \( y(t) \))

*See Ref. 811025/6232 for the frequency domain equality associated with the LQ problem.
\[ G_{\text{FOL}}(s) \overset{\Delta}{=} C(sI-A)^{-1}L \]  
(3-6)

\[ G_{\text{FOL}}^H(s) \overset{\Delta}{=} L'(-sI-A')^{-1}C' \]  
(3-7)

Then the following equality holds

\[
[I+G_{\text{KF}}(s)]^T[I+G_{\text{KF}}(s)]^H = \\
= \Theta + G_{\text{FOL}}(s)G_{\text{FOL}}^H(s)
\]  
(3-8)

If

\[ \Theta = \mu I; \quad \mu > 0 \]  
(3-9)

then Eq. (3-8) reduces to

\[
[I+G_{\text{KF}}(s)][I+G_{\text{KF}}(s)]^H = I + \frac{1}{\mu} G_{\text{FOL}}(s)G_{\text{FOL}}^H(s)
\]  
(3-10)

3.4 Guaranteed Robustness Properties

The KF enjoys the same type of robustness properties as the LQ regulator. The following properties are valid if

\[ \Theta = \text{diagonal matrix} \]  
(3-11)

From the frequency domain equality (3-8) we deduce the inequality

\[ [I+G_{\text{KF}}(s)][I+G_{\text{KF}}(s)]^H \geq I \]  
(3-12)

From the definition of singular values we then deduce that

\[ \sigma_{\min}[I+G_{\text{KF}}^{-1}(s)] \geq 1 \]  
(3-13)

\[ \sigma_{\min}[I+G_{\text{KF}}^{-1}(s)] \geq \frac{1}{2} \]  
(3-14)
3.5 Optimal KF Root Locus

The root-square locus techniques* for LQ regulators can be readily adapted to root-locus studies involving the closed-loop poles of the Kalman filter.

4. THE KALMAN FILTER AS A TRIVIAL CONTROL PROBLEM

(to be written).

*See Ref. 811028/6232
Lecture Notes

on

THE ACCURATE MEASUREMENT KALMAN FILTER PROBLEM

Written by

Professor Michael Athans

April 24, 1985

REF. NO. 850425/6232

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THE ACCURATE MEASUREMENT KALMAN FILTER PROBLEM

0. SUMMARY

In this note we summarize the properties of the Kalman Filter (KF) problem when the intensity of the sensor noise approaches zero. In a mathematical sense this is the "dual" of the so-called "cheap-control LQR problem. The results are fundamental to the Loop Transfer Recovery (LTR) method applied at the plant input.

1. PROBLEM DEFINITION

Consider the stochastic LTI system

\[ \dot{x}(t) = A x(t) + L \xi(t) \]  
\[ y(t) = C x(t) + \theta(t) \]

(1.1)  
(1.2)

We assume that the process noise \( \xi(t) \) is white, zero-mean, and with unit intensity, i.e.

\[ E(\xi(t)\xi'(t)) = I \delta(t-\tau) \]

(1.3)

We also assume that the measurement noise \( \theta(t) \) is white, zero-mean and with intensity indexed by \( \mu \), i.e.

\[ E(\theta(t)\theta'(t)) = \mu I \delta(t-\tau) \]

(1.4)

Definition 1.1: What we mean by the accurate measurement KF problem is defined by the limiting case

\[ \mu \to 0 \]  

(1.5)

corresponding to essentially noiseless measurements.
Under the assumptions that $[A, L]$ is stabilizable and that $[A, C]$ is detectable we know that the KF is a stable system and generates the state estimates $\hat{x}(t)$ by

$$\dot{\hat{x}}(t) = [A - H_{\mu} C] \hat{\hat{x}}(t) + H_{\mu} y(t)$$  

(1.6)

where we use the subscript $\mu$ to stress the dependence of the KF gain matrix $H_{\mu}$ upon the parameter $\mu$.

We recall that $H_{\mu}$ is computed by

$$H_{\mu} = \frac{1}{\mu} \Sigma_{\mu} C'$$  

(1.7)

where the error covariance matrix $\Sigma_{\mu}$, also dependent upon $\mu$, is calculated by the solution of the FARE:

$$0 = A \Sigma_{\mu} + \Sigma_{\mu} A' + L L' - \frac{1}{\mu} \Sigma_{\mu} C' C - \Sigma_{\mu}$$  

(1.8)

We seek insight about the limiting behavior of both $\Sigma_{\mu}$ and $H_{\mu}$ as $\mu \to 0$.

2. THE MAIN RESULT

In this section we summarize the main result in terms of a theorem.

Theorem 2.1: Suppose that the TFM from the white noise $\mu(t)$ to the output $y(t)$ for the system (1.1), (1.2), i.e. the TFM

$$\hat{w}(s) \triangleq C(sI - A)^{-1} L,$$  

(2.1)

is minimum phase. Then,

$$\lim_{\mu \to 0} \Sigma_{\mu} = 0$$  

(2.2)
and
\[ \lim_{\mu \to 0} \mu H = L \quad \text{if} \quad W'W = 1 \]  

(2.3)

**Proof:** This is theorem 4.14 in Kwakernaak and Sivan, Ref. [1], pp. 370-371.

**Remark 2.1:** It can be shown that the minimum phase of \( H(s) \), given by eq. (2.1), is both a necessary and a sufficient conditions for the limiting properties given by eqs. (2.2) and (2.3).

**Remark 2.2:** The implication of eq. (2.2) is that in the case of exact measurements upon a minimum phase plant, the KF yields exact state estimates, since the error covariance matrix is zero. This assumes that the KF has been operating upon the data for a sufficiently long time so that initial transient errors have died out.

**Remark 2.3:** For a non-minimum phase plant
\[ \lim_{\mu \to 0} \frac{\Sigma}{\mu} \neq 0 \]  

(2.4)

Hence, perfect state estimation is impossible for non-minimum phase plants.

**Remark 2.4:** The limiting behavior (with \( L = B \)) of the Kalman Filter gain
\[ \lim_{\mu \to 0} \sqrt{\mu} H = B \quad \text{if} \quad W'W = 1 \]  

(2.5)

is the precise dual of the limiting behavior of the LQ control gain
\[ \lim_{\rho \to 0} \sqrt{\rho} G = W C \quad \text{if} \quad W'W = 1 \]  

(2.6)

for the minimum phase plant
\[ G(s) = C(sI-A)^{-1}B \]  

(2.7)
The relation (2.5) has been used by Doyle and Stein [2] to apply the LTR method at the plant input, while eq. (2.6) has been used by Kwakernaak [3] to apply the LTR method at the plant output (see also Kwakernaak and Sivan [1], pp. 419-427).

3. REFERENCES


THE KALMAN FREQUENCY DOMAIN EQUALITY

Up to this point we have established that the LQR method with full state feedback automatically generates stable closed-loop systems. Closed-loop stability of the nominal plant is important but not enough. One must understand the frequency domain implications to understand its corresponding loop-shaping properties, crossover frequency determination, robustness in terms of SV's.

The frequency domain properties of LQR designs depend upon the so-called K-Frequency Domain Equality. Assume,

\[ \dot{x}(t) = A x(t) + B u(t) \]  \hspace{1cm} (1)

\[ y(t) = C x(t) \]  \hspace{1cm} (2)

and also, \([A, B]\) is stabilizable \[ (3) \]
\([A, C]\) is detectable \[ (4) \]

Let \( Q = \Sigma \Sigma^T \) be the state weighting matrix \( R = R^T > 0 \) be the control weighting matrix in the LQ cost functional:

\[ J = \int_0^\infty [x'(t) Q x(t) + u'(t) R u(t)] \, dt \]  \hspace{1cm} (5)
Then, the optimal control is:
\[ u(t) = -Gx(t) \quad (6) \]
where
\[ G = R^{-1}BK \quad (7) \]
and
\[ K = K' > 0 \quad is \quad the \quad solution \quad of \quad ARE \]
\[ A'K + KA + C - KBKR^{-1}B'K = 0 \quad (8) \]

**Block Diagram for the LQR.**

For this loop, the \( LTI \) matrix is:
\[ G_{LQR}(s) = G(sI - A)^{-1}B \quad (9) \]
& the return difference is
\[ [I + G_{LQR}(s)] \quad (10) \]

The \( \mathcal{H}_2 \) freq. Domain eq. relates (10) with the open loop \( TM \)
\[ G_{ol}(s) \] given by:
\[ G_{ol}(s) = C(sI - A)^{-1}B \quad (11) \]
Statement of the Kalman Equality

The following equality among transfer matrices is true:

\[
\begin{bmatrix}
I + G_{LR}(-s)
\end{bmatrix} R \begin{bmatrix}
I + G_{LR}(s)
\end{bmatrix} = R + G_{OL}(-s) G_{OL}(s),
\]

which establishes a valuable relation between the open-loop TM, \( G_{OL}(s) \), and the closed-loop LQR return difference TM.
Kalman Freq. Domain Eqauality:

\[
\begin{align*}
G_{KF}(s) & \approx \frac{1}{\sqrt{\rho}} \quad G_{FOL}(s) \approx \frac{1}{\sqrt{\rho}} \quad C \ (sI - A)^{-1} \ L \ ; \quad \Xi = \Xi^T \\
& \quad \Theta = \Theta^T \end{align*}
\]

\[
\begin{align*}
G_{LQ}(s) & \approx \frac{1}{\sqrt{\rho}} \quad G_{OL}(s) \approx \frac{1}{\sqrt{\rho}} \quad M \ (sI - A)^{-1} \ B \ ; \quad \Omega = \Omega^T \\
& \quad \Theta = \Theta^T \quad R = sI
\end{align*}
\]
**Motivation**

LTR followed investigations related to guaranteed robustness properties of LQ regulators & TF optimal stochastic estimators. During that time, it was kept unknown whether LQG regulators had any guaranteed robustness properties. However, it was demonstrated that arbitrary LQG regulators do not have any guaranteed robustness properties.

The LTR method was the outcome of subsequent investigation directed towards possible ways of improving the robustness properties of LQG regulators. However, later on it was realized that the LTR method has little to do with the LQG regulators. So it is appropriate to proceed LTR in a more general framework using MBC's.
MODEL BASED COMPENSATORS (MBC) AND THE LOOP TRANSFER RECOVERY (LTR) METHOD

The objective here is to describe the so-called loop Transfer Recovery (LTR) method for the class of feedback control system designs which use a MBC.

It is important to note that this LTR method is only applicable to SISO or MIMO plants which are minimum phase, i.e. the transmission zeros of the plant are strictly inside the left-half s-plane.

I. SETTING THE STAGE

Assume the following structure for feedback:

\[ \begin{array}{c}
\dot{x}(t) = A x(t) + B u(t) \\
y(t) = C x(t)
\end{array} \]  \hfill (II.1)

1. Define

\[ F(s) = (s I - A)^{-1} \]  \hfill (II.2)

So,

\[ G(s) = c (s I - A)^{-1} B = c F(s) B \]  \hfill (II.3)
**Assumptions:**

1. The plant is square, i.e., \# of inputs is equal to \# of outputs.

2. Plant is: (a) stabilizable (or controllable)  
   (b) detectable (or observable)

3. Plant model is strictly minimum phase; i.e., all of the transmission zeros of \( G(s) \) are strictly in the LHP.

**MBC Compensator:**

The MBC transfer function \( K(s) \) is:

\[
K(s) = G (sI - A + BG + HC)^{-1} H
\]

(II. 4)

where \( G \) is the control gain matrix, and \( H \) the filter gain matrix. Since \( A, B, C \) are fixed, the gain matrices \( G \) and \( H \) represent the "design" parameters for the MBC.

The stabilizability & detectability assumptions guarantee that there exist matrices \( G \) & \( H \) such that the closed-loop system is unconditionally stable. In fact, in view of the separation property of MBC designs,
the closed-loop poles \( \lambda_i[A-Bc] \) and \( \lambda_i[A-Hc] \) have the property:
\[
\text{Re} \lambda_i[A-Bc] < 0 \\
\text{Re} \lambda_i[A-Hc] < 0
\]

Note: As long as we guarantee closed-loop stability, the choice of the MBC design parameter matrices (i.e. \( G \) and \( H \)) will determine what kind of an MBC we have and will determine the loop matrices:

\[
I(s) = G(s)K(s) \quad \text{(III.5)}
\]
(Loop Transfer Matrix at Plant Output)

\[
I'(s) = K(s)G(s) \quad \text{(III.6)}
\]
(Loop Transfer Matrix at Plant Input).

As we'll see later these will determine the performance and robustness characteristics of the closed-loop design.

The two LTR methods

Given a MIMO system that has been there are two distinct LTR methods.

**LTR at Plant Output:** Fix \( H \), vary \( G \).

In this method, we fix the feedback gain \( H \) and we appropriately select the control gain matrix \( G \) in the MBC.
This impacts the behavior of the loop TM at the plant output, i.e.,

\[ I'(s) = G(s)K(s) \quad (II.7) \]

**LTR at Plant Input:** Fix \( G \) & vary \( H \).

We now fix the control gain matrix \( G \) & appropriately select the filter gain matrix \( H \). This impacts \( I'(s) \); i.e.,

\[ I'(s) = K(s)G(s) \quad (II.8) \]

It is worth mentioning that once we fix \( G \), \( H \) should be adjusted to maintain closed-loop stability.

**Remark:** These two methods of executing LTR are related; actually they are dual from a mathematical point of view.

We are going to present the LTR method in which \( H \) is fixed & \( G \) is adjusted. This version is consistent with shaping the SV's of \( I'(s) \)

\[ I'(s) = G(s)K(s) \quad (II.9) \]

i.e. (a) Command following & output disturbance rejection of
(b) Robustness with modeling errors reflected at the plant output.
III. THE MBC/LTR Method at the Plant Output

III.1. Problem Formulation

Suppose that in our MBC we have selected a filter gain matrix \( H \) such that:
\[
\text{Re}[A - H \Sigma] < 0 \quad (\text{III.1})
\]

How \( H \) is selected is immaterial; all that matters is our assumption that \([A, H]\) is detectable guarantees to us that we can find "small" filter gain matrices such that (III.1) is true.

Remark: Eventually, in the LQG/LTR procedure, the filter gain matrix \( H \) will be selected by solving a special KF optimal estimation problem.

Once the filter gain \( H \) is selected (which has nothing to do with LTR) the only other choice left to the designer is the selection of the control matrix \( G \).

Important Basic Idea: The fundamental idea behind this LTR method is to generate a parametrized family of control gain matrices \( G_p \) such that:

\[(\text{III.2}) \quad \begin{align*}
(a) & \text{ The scalar real parameter } p \text{ varies from } 0 < p < \infty. \\
(b) & \text{ The parametrization of } G_p \text{ has the property that } \\
& \lim_{p \to 0} G_p \to W C
\end{align*} \quad (\text{III.3})\]
where $E$ is the plant-model matrix appearing in the output-to-state time-domain model, and where $W$ is a unitary matrix, i.e.

$$WW = I \quad (III.4)$$

**Remarks:**
1. It should not be obvious at this stage why such a parametrization of the control gain matrix $G_p$ should be considered.
2. It should be even less obvious how to select matrices $G_p$ that have the required property and at the same time maintain closed-loop stability, i.e.

$$Re \gamma [A - B G_p] < 0 \quad (III.5)$$

holds as $p$ approaches its limit ($p \to 0$).

Now we discuss the cheap-control minimum phase optimal LQ regulator problem which will be helpful from a design viewpoint, even though theoretically has very little significance.

### 2. The LQ Cheap Control Problem

Consider the following LQ optimal regulator problem.

**Minimize,**

$$J = \int [y(t) y(t) + p \nu'(t) y(t)] dt; \quad p > 0 \quad (III.6)$$

subject to the dynamic constraints

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow f(t) = E x(t) \quad (III.7)$$

We know that the LQ control law is:
\[ u(t) = -G_p \times (t) \] 

where the control gain \( G_p \) is given by

\[ G_p = \frac{1}{p} B' K_p \]  

and \( K_p \) is given by the control algebraic Riccati equation

\[ -K_p A - A' K_p - C'C + \frac{1}{p} K_p B B' K_p = 0 \]

Note that we have emphasized the dependence of the appropriate quantities upon the control weighting parameter \( p \), which appears in the quadratic cost functional of (III.8).

Now we present these theorems.

**Theorem 1.** (Prove in H. Kunkel and H. R. S. Islam, pp. 306-312).
Under the controllability assumptions and the minimum phase assumption, 
\[ \lim_{p \to 0} K_p \to 0 \]  

The minimum phase property of (III.6) is both a necessary and sufficient condition for the limiting behavior.

**Theorem 2.** Consider the cheap LD-problem, defined as \( p \to 0 \). Suppose that the same assumptions as for Theorem 1 hold. Then,
\[
\lim_{\rho \to 0} \sqrt{\rho} G_{\rho} \to W C; \quad W W = I
\]

Remark: (1) The proof of theorem 2 demonstrates that the desired structure (III.12) arises from the solution of the cheap L2 regulator problem with the minimum phase assumption.

(a) In order to carry out numerically designs that exploit the LTR method in conjunction with MAC-based feedback designs, one must guarantee that \( \text{Re} \, \gamma(t) \leq \rho \) as \( \rho \) is adjusted. However, we know that LQ designs are guaranteed to be stable under the -ability assumption, and therefore the LQ control problem offers as a "fail-safe" recipe for the parametrization required by the LTR method. It should be emphasized that this method only works for minimum phase plants.

(b) There is another way of computing the desired limiting behavior of the control gain. Consider the following quadratic cost function:

\[
J_1 = \int_{0}^{\infty} \left[ 2 y'(t)y(t) + u'(t)u(t) \right] dt; \quad \rho > 0
\]

or

\[
J_1 = q \int_{0}^{\infty} \left[ y'(t)y(t) + \frac{1}{q} u'(t)u(t) \right] dt
\]

The optimal control for \( J_1 \) is identical for the optimal control for

\[
J = \int_{0}^{\infty} \left[ y(t)y(t) + p y'(t)u'(t) \right] dt \quad \text{with} \quad p = \frac{1}{q}
\]
So, if we allow $q \to \infty$ we get the cheap LQ control problem.

To summarize: We have demonstrated how to construct control gain matrices $G_p$ with the property that

$$\lim_{p \to \infty} \sqrt{p} G_p \to WC; \quad WW' = I,$$

which, as we will see, is a key property that is used to develop the LTR method.

### III.3 Development of the LTR method

We can now develop the LTR method for MBC based designs. We first consider fixed filter gain $H$ and parametrized control gains $G_p$ that have the property

$$\lim_{p \to \infty} \sqrt{p} G_p \to WC; \quad WW' = I.$$

The following lemma (w/o proof) summarizes the behavior of this class of MBC/LTR compensators.

**Lemma 1**: Consider the class of MBC's with the transfer function matrix $K_p(s)$:

$$K_p(s) = G_p \left( sI - A + BG_p + HC \right)^{-1} H$$

such that:

$$\text{Re} \, \varpi_i [A-KC] < 0$$

$$\text{Re} \, \varpi_i [A-BG_p] < 0, \text{ all } 0 < p < \infty$$
and such that
\[ \lim_{p \to 0} \bar{P} \bar{G}_p \to WC; \quad \bar{W} \bar{W} = I \] (III.16)

Then,
\[ \lim_{p \to 0} K_p(s) \to \left[ C \left( sI - A \right)^{-1} B \right]^{-1} \left( sI - A \right)^{-1} H \] (III.17)

where the convergence is pointwise in \( s \).

### III.4 The main Result: The MBC/LTR Method

We can now state the LTR method as applied to MBC-based MIMO feedback designs.

Consider an MBC-based MIMO design as shown above where \( K_p(s) \) denotes an MBC that satisfies the above lemma and \( G(s) \) a given plant represented by \([G(s)], [\]s\).

Let's break the loop at the plant output (or at the error signal). Then, the loop TF is

\[ T^o_p(s) = G(s) \cdot K_p(s) \] (III.18)

where the dependence of \( T^o_p(s) \) on \( p \) is emphasized.

So, the MBC-based design and its performance
robustness properties are completely specified by the following system:

Now, for any filter gain matrix \( H \), we define a filter-loop transfer matrix, denoted by \( G_F(s) \), as:

\[
G_F(s) = \mathcal{C} (sI - A)^{-1} \mathcal{H} = \mathcal{C} \Phi(s) \mathcal{H} \\
(III.19)
\]

Also recall that:

\[
G(s) = \mathcal{C} (sI - A)^{-1} \mathcal{B} = \mathcal{C} \Phi(s) \mathcal{B} \\
(III.20)
\]

Then the asymptotic property of \( \mathcal{MFC} \) can now be written as:

\[
\lim_{\rho \to 0} K_F(s) \rightarrow G^{-1}(s) G_F(s) \\
(III.21)
\]

Therefore we can deduce that:

\[
\lim_{\rho \to 0} I_F(s) = G(s) G^{-1}(s) G_F(s) \\
(III.22)
\]

or

\[
\lim_{\rho \to 0} I_F^*(s) = G_F(s) = \mathcal{C} \Phi(s) \mathcal{H} \\
(III.23)
\]

Therefore the limiting behavior of the \( \mathcal{MFC} \)-based design approaches that of the filter-loop shown as:
The filter loop.

Remarks: (1) It is clear now that the limiting MBC compensator inverts the plant \( G(z) \) and yields a new loop \( T_{M} \frac{G(f(s))}{G(z)} \). Hence, the MBC/LTR design will behave like the closed-loop system shown above, at least as far as performance (command following & disturbance rejection) is concerned. As far as robustness is concerned, as long as we reflect modeling errors or the plant output, the robustness characteristics of the MBC/LTR design will also be closely approximated by those defined by the filter loop.

(2) If the filter loop is designed "stupidly," then the MBC/LTR design will be bad. Conversely, if the system filter loop is designed well, then the modern design will be good. Therefore, if the designer wishes to use the MBC/LTR method, he must be able to easily calculate (by whatever method) filter gain matrices \( H \) that not only guarantee nominal stability of the filter loop, but, at the same time, this filter loop must be characterized by:
(a) good crossover properties,
(b) good command-following & disturbance-rejection properties, and
(c) good robustness to unmodeled dynamics.

As we'll see, choosing the filter gain H by Kalman filtering methods helps the designer achieve the above objectives.

(3) The word "Loop Transfer Recovery (LTR)" reflects the implications of the above remarks. It is assumed that a good "loop transfer" matrix \( G_c(s) \) has been designed by the engineer. This good loop is then "recoved" by the LBC/LTE method.

(4) It was stated that the LTR limiting process is pointwise in \( s \). This means that for the same value of \( p \) and two different values of the complex variable \( s \), say \( s_1 \) and \( s_2 \), the difference \( \| \tilde{L}(s_1) - G^{-1}(s_1) G_c(s_1) \| \) and \( \| \tilde{L}(s_2) - G^{-1}(s_2) G_c(s_2) \| \) are not the same. Similarly for the \( \tilde{K}(s) \). At least for minimum phase plants, results show that the SV's of \( \tilde{K}(s) \) approach the SV's of \( G_c(s) \) faster in the low frequency region as \( p \) is decreased further and further.

(5) The limiting formulas that we employed can cause some confusion about the dimensionality of \( \tilde{K}(s) \).
Suppose that the open-loop plant $G(s)$ has $n$ poles and $m$ zeros ($m \leq n-1$). The transfer function is a strictly proper $m$th order system, so that $E(s)$ also has $n$ poles & $m$ zeros ($m \leq n-1$). Hence the $I_p(s)$ has $2n$ poles & $2m$ zeros ($2m \leq 2n-2$).

On the other hand, the filter transfer matrix $G_f(s) = \frac{C(s)}{D(s)}$ has only $n$ poles. The selection of $q$ will determine the # of locations of the zeros of $G_f(s)$; however, it will be strictly proper now, since the LHS of the limiting relation has a $2m$th order system, while the RHS has only an $m$th order, this may be misleading. However, it should be remembered that we only have pointwise convergence.

(b) Let's suppose that $G(s)$ is minimum phase but, otherwise unrestricted, i.e. its may be open-loop unstable. Then, if $m$ represents the # of open-loop zeros of $G(s)$, $m \leq n-1$, it follows that $m$ (out of $n$) poles of the $I_p(s)$ must cancel (approximately) the $m$ open-loop zeros of $G(s)$. For min-phase plants, this is "legal", since we are cancelling poles & zeros in the LHP.

(f) Now suppose that the open-loop plant $G(s)$ has one or more non-minimum phase zeros. In principle, it is still possible to have a plant inversion by having the appropriate # of compensation poles cancel the RHP
However, from a practical point of view, this is nonsense, since the physical closed-loop system is unstable. The result is that we cannot rely upon MBC/LTR to recover, in general, arbitrary loop transfer matrices \( g_\ell(s) \) whenever \( G(s) \) is a non-minimum phase plant.

Actually, if we use cheap control LD procedure to compute the gain matrices \( G_p \), as \( P \rightarrow 0 \), we will not "recover" arbitrary filter-loop transfer \( \ell \). \( g_\ell(s) \), whenever \( G(s) \) has non-minimum phase zeros at the "wrong" frequencies.

**IV. BOTTOM LINE**

Now we summarize the important results of MBC/LTR design.

**Lesson #1:** If the open-loop plant is non-minimum phase, the MBC/LTR design method has to be used with caution. Whether the MBC/LTR gives good designs depends strongly on the location of the non-minimum phase zeros of the \( G(s) \) as compared to the crossover behavior of the filter loop \( \ell \). \( G_\ell(s) \).

**Lesson #2:** If \( G(s) \) is minimum phase then MBC/LTR can be used in a routine manner. It is recommended that the MBC gain matrix \( G_p \) be computed for \( P \) close to zero using the solution of the cheap control LD regulator problem.
Lesson #3: For minimum phase plants the design procedure reduces to two steps:

**Step 1:** Given the open-loop plant, i.e. given $A(s)$, find a good filter gain matrix $H$ such that the loop transfer matrix $G_f(s) = F(s)H$ and therefore, the target loop, has the required

(a) command-following & disturbance rejection properties (i.e. good performance);

(b) stability-robustness properties, when the modeling errors are reflected at the plant output;

(c) good crossover frequencies, i.e., $SV's$ of $G_{f1}(s)$ cross the 0db line with slopes near -20 db/dec.

**Step 2:** Use $H$ in the MBC and apply the MBC/LQR method. Compute the control gain matrix $G_p$ using the solution for the cheap-control LQ regulator problem.

It should be obvious that now the major concentration of the design has shifted (for minimum phase plants) to easy ways of calculating the filter gain $H$, such that $G_f(s) = \frac{1}{I + G(s)}$, and $[I + G(s)]$ have "good" loop-shapes.
The goal is to use the KF optimal estimation problem in order to accomplish the objective of finding a target loop.

References:


SHAPING THE SINGULAR VALUES OF THE KALMAN FILTER LOOP TRANSFER MATRIX

We now present explicit formulas that can be used to select the design parameters of the KF problem so that the KF LTPM, $G_{KF}(s)$, has desired freq. domain properties as quantified by the shapes of the SV's $\sigma_i [G_{KF}(j\omega)]$ vs. frequency.

It is important to realize that what is presented here is important when the KF loop represents the target design, in the LQG/LTR method. We know that for minimum phase plants the LTR will recover the KF loop and its properties.

The tricks presented here depend upon the proper interpretation of the KF filter frequency domain equality ($KFDE$) and the approximations valid at low & high frequencies.

To. Definition: The KF loop hinges upon the design of the KF LTPM $G_{KF}(s)$ given by:

$$G_{KF}(s) = C \Phi(s) H$$ \hspace{1cm} (1.1)

where

$$\Phi(s) = (sI - A)^{-1}$$ \hspace{1cm} (1.2)

and $H$ being the KF gain matrix.
Recall that the KF gain matrix is computed by:

$$ H = M \Sigma C' $$

where $$ \Sigma = \Sigma' \Sigma > 0 $$ is the solution of FARE:

$$ A \Sigma + \Sigma A' + \Sigma B' - \frac{1}{\mu} \Sigma C \Sigma = 0 \quad (I.4) $$

The matrix $$ L $$ and the scalar $$ \mu $$ are the design parameters. Once $$ L $$ and $$ \mu $$ are selected, $$ \Sigma $$ can be calculated by (I.4) and $$ H $$ by (I.3). Consequently, the KF LFM is fixed and all stability, robustness and performance properties of the KF loop are determined.

Since the target KF loop must meet the desired specs in the $\mathcal{L}_\infty$/CTR method, and since such specs are imposed upon the loop SV's in the frequency domain, it is natural that we try to determine $$ L $$ using frequency domain tools.

It is now convenient to define an auxiliary square LFM denoted by $\tilde{G}_{ol}(s)$ as:

$$ \tilde{G}_{ol}(s) \equiv C \tilde{F}(s) L $$

Note that the poles of $\tilde{G}_{ol}(s)$ are the plant model poles. The design parameter $$ L $$ fixes the number, location and directions of the $\mathcal{L}_\infty$ transmission zeros of $\tilde{G}_{ol}(s)$. 
The KFFDE states that:

\[
\left[ I + G_{KF}(s) \right]\left[ I + G_{KF}(-s) \right]' = I + \frac{1}{\mu} G_{FOL}(s) G_{FOL}'(-s) \]  \hspace{1cm} (I.6)

Far from crossover and in particular for \( s = j\omega \to 0 \) and at \( s = j\omega \to \infty \) one can deduce the following approximation:

\[
G_{KF}(s) \approx \frac{1}{\sqrt{\mu}} G_{FOL}(s) \approx \frac{1}{\sqrt{\mu}} \left( sI - A \right)^{-1} \]  \hspace{1cm} (I.7)

This approximation provides insight as to how to select \( \mu \) and \( \ell \).

III. Matching Singular Values at DC

Assume that \( A' \) exists, so that the open-loop plant has no natural integrators.

As \( s = j\omega \to 0 \) we have the following limiting behavior:

\[
\lim_{s \to 0} (sI - A)^{-1} = -A' \]  \hspace{1cm} (II.1)

Hence,

\[
\lim_{s \to 0} G_{FOL}(s) = -\frac{C}{\ell} A'^{-1} \]  \hspace{1cm} (II.2)
which means that as $s = j \omega \to 0$

$$\sigma_{\ell} [G_K(s)] \approx \frac{1}{\sqrt{\mu}}$$

Let's suppose that at low freq. we want to make the SV's of $G_K(s)$ approximately the same. Then, we can select $L$ such that

$$-C A' L = I$$  \hspace{1cm} (II.4)

If this is true, then as $s = j \omega \to 0$

$$\sigma_{\ell} [G_K(s)] \approx \frac{1}{\sqrt{\mu}}$$  \hspace{1cm} (II.5)

so that $\mu$ can be adjusted to control the size of the low frequency mino loop gain.

Two way of selecting $L$ such that (II.4) holds are:

(A)  \hspace{1cm} $$L = -C' (C A^{-1} C')^{-1}$$  \hspace{1cm} (II.6)

(B)  \hspace{1cm} $$L = -A C' (C C')^{-1}$$  \hspace{1cm} (II.7)
III. Matching SV's at High Frequencies

As $j\omega \to \infty$, we have the following limiting behavior

$$\lim_{s \to \infty} (sI - A)^{-1} = \frac{I}{s} \quad \text{(III.1)}$$

Hence,

$$\lim_{s \to \infty} G_{FDL}(s) = \frac{CL}{s} \quad \text{(III.2)}$$

Therefore at $s = j\omega \to \infty$,

$$\sigma_i \left[ G_{KF}(j\omega) \right] \approx \frac{1}{\omega} \frac{1}{\sqrt{j\omega}} \sigma_i \left[ CL \right] \quad \text{(III.3)}$$

which demonstrates that the SV's of the KF LTM roll off at $-20\text{dB/dec}$.

Again, the parameter $I$ can be chosen to control the loop gain at high frequencies, influencing the loop crossover frequency. If we wish the SV's to be identical at high freqs., then,

$$CL = I \quad \text{(III.4)}$$

This can be accomplished if:

$$T = C' (CC')^{-1} \quad \text{(III.5)}$$
One alternate method that will allow more degrees of freedom is:

\[ L = NM \]  \hspace{1cm} (III.6)

where \( N \) is \( nxn \) and \( M \) is \( nxm \). Then we want

\[ CNM = I \]  \hspace{1cm} (III.7)

So, we select \( M = C'(CNC')^{-1} \)  \hspace{1cm} (III.8)

so that,

\[ L = NC'(CNC')^{-1} \]  \hspace{1cm} (III.9)

Different choices of \( N \) should result into different crossover and low frequency S/V shapes. However, currently no systematic way exist to do so.

II. Adding Integrators in the KF loop

If the open-loop plant has no integrators (\( A^r \) exists) then the KF loop will exhibit steady-state errors to constant command and/or disturbance inputs. Therefore, the LQG/LTR loop will also have similar behavior.

It is common that specs include zero steady-state errors to constant command and/or disturbance.
inputs. Hence, we must modify the way the KR
loop is designed. This can be done by adding
integrators at the plant input.

We start with the plant model described by
the state equations

\[
\begin{align*}
\dot{x}_p(t) &= A_p x_p(t) + B_p u_p(t) \\
\frac{d}{dt}p(t) &= C_p x_p(t)
\end{align*}
\]  \tag{IV.1}

and the TFM

\[
G_p(s) = C_p (sI - A_p)^{-1} B_p \tag{IV.2}
\]

We assume that $A_p^*$ exists so that there are
no natural integrators.

Now we augment the plant by introducing int
integrators at the plant control channels. That is,

\[
\dot{u}_p(t) = \hat{y}(t) \tag{IV.3}
\]

with

\[
G_a(s) = \frac{I}{s} \tag{IV.4}
\]

\[
\begin{array}{c}
H(s) \xrightarrow{\frac{I}{s}} \frac{I}{s} \\
\xrightarrow{G_a(s)} \\
\xrightarrow{G_p(s)} \\
\xrightarrow{GP(s)}
\end{array}
\]
If we define,
\[ g_p(s) = G(s) u(s) \]  \hspace{1cm} (IV.5)
then,
\[ G(s) = G_p(s) G_a(s) = G_p(s I - A_p)^{-1} B_p \frac{I}{s} \]  \hspace{1cm} (IV.6)

To compute the state-space description of augmented system, define:
\[ \dot{x}(t) = [u_p(t) \ x_p(t)] \hspace{1cm} (IV.7) \]
where \( x(t) \in \mathbb{R}^{n+m} \). Then,
\[ \begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
\dot{y}_p(t) &= C x(t)
\end{align*} \hspace{1cm} (IV.8) \]
where,
\[ A = \begin{bmatrix} 0 & 0 \\ B_p & A_p \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \end{bmatrix} \hspace{1cm} (IV.9) \]
\[ C = \begin{bmatrix} 0 & G_p \end{bmatrix} \hspace{1cm} (IV.10) \]

Note that we have preserved the significance of the plant output \( y_p(t) \).

Now, we use the augmented dynamics (IV.8) when designing the KF loop.
The KFDE holds for the augmented system. So using $A, B, \theta \in$ we can still use the approximation,

$$\underline{G_{KF}}(s) \approx \frac{1}{\sqrt{\mu}} \underline{G_{OL}}(s) \quad (IV.11)$$

far from crossover.

In general $\underline{G_{OL}}(s)$ is defined by:

$$\underline{G_{OL}}(s) = \underline{C} \left( sI - \underline{A}_{p} \right)^{-1} \underline{L} \quad (IV.12)$$

So,

$$\underline{G_{OL}}(s) = \left[ \begin{array}{c} 0 \\ \underline{C} \end{array} \right] \left[ \begin{array}{cc} sI & 0 \\ -\underline{B}_{p} & sI - \underline{A}_{p} \end{array} \right]^{-1} \left[ \begin{array}{c} \underline{L}_{L} \\ \underline{L}_{q} \end{array} \right] \quad (IV.13)$$

In order to understand the low and high frequency behavior of $\underline{G_{OL}}(s)$ we should focus our attention on $(sI - \underline{A}_{p})^{-1}$. We have,

$$(sI - \underline{A}_{p})^{-1} \equiv \Phi(s) = \left[ \begin{array}{cc} \frac{I}{s} & 0 \\ \left(sI - \underline{A}_{p}\right)^{-1} & \left(sI - \underline{A}_{p}\right)^{-1} \end{array} \right] \quad (IV.14)$$
Now we examine the behavior of $\Phi(s)$, given by equation (IV.14) at low frequencies ($s \to 0$) and high frequencies ($s \to \infty$).

**Low Frequency Limits**

$$\lim_{s \to 0} \Phi(s) = \begin{bmatrix} \frac{I}{s} & 0 \\ \frac{A_p B_p}{s} + A_p^{-1} & -A_p^{-1} \end{bmatrix} \quad (IV.15)$$

**High Frequency Limits**

$$\lim_{s \to \infty} \Phi(s) = \begin{bmatrix} \frac{I}{s} & 0 \\ \frac{B_p}{s^2} & \frac{I}{s} \end{bmatrix} \quad (IV.16)$$

We now have all we need to investigate the $G_{FO}(s)$ behavior at both low and high frequencies.

**IV.1 Mathing Singular Values at Low Frequencies**

We can now conclude that:

$$\lim_{s \to 0} G_{FO}(s) = \begin{bmatrix} 0 & Cp \\ \frac{I}{s} & 0 \end{bmatrix} \begin{bmatrix} \frac{I}{s} & 0 \\ \frac{A_p^{-1} B_p}{s} & -A_p^{-1} \end{bmatrix} \begin{bmatrix} L \nu \n \end{bmatrix} \quad (IV.17)$$

So,

$$\lim_{s \to 0} G_{FO}(s) = \frac{C_p A_p^{-1} B_p L \nu}{s} - C_p A_p^{-1} \nu \quad (IV.18)$$
As $s \to 0$, the first term dominates. Therefore,

$$\lim_{s \to 0} G_{fo}(s) = \frac{\mathcal{C}_p A_p^{-1} B_p}{s} \quad (11.19)$$

At low frequencies:

$$6: \left[ G_{kr}(j\omega) \right] = \frac{1}{\sqrt{\nu}} \delta_i \left[ G_{fo}(j\omega) \right] \quad (11.20)$$

to match the SV's. at DC we want

$$\mathcal{C}_p A_p^{-1} B_p L_i = I \quad (11.21)$$

This can be accomplished if:

$$L_i = \frac{I}{\mathcal{C}_p A_p^{-1} B_p} \quad (11.22)$$

The above equation states that $L_i$ is the inverse of the DC plant gain matrix ($\mathcal{C}_p^{-1}$).

Note that all SV's of $G_{kr}(j\omega)$ are decreasing at $-20$ dB/dec at DC, as required by the zero steady-state error spec.
### III.2 Matching Singular Values at High Frequencies

Now we can show how the left-over parameter $L_1$ can be selected so that we match the SV's of $G_{RF}(j\omega)$ at high frequencies.

We see that

$$\lim_{s \to \infty} G_{\text{fol}}(s) = \begin{bmatrix} 0 & C_p \end{bmatrix} \begin{bmatrix} \frac{i}{s^2} & 0 \\ \frac{B_p}{s^2} & \frac{i}{s} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \quad (IV.23)$$

or

$$\lim_{s \to \infty} G_{\text{fol}}(s) = \frac{C_p B_p}{s^2} L_1 + \frac{C_p}{s} L_2 \quad (IV.24)$$

Clearly as $s \to \infty$, the second term dominates and,

$$\lim_{s \to \infty} G_{\text{fol}}(s) = \frac{C_p L_1}{s} \quad (IV.25)$$

As, $j\omega \to \infty$

$$\delta_i \left[ G_{RF}(j\omega) \right] \approx \frac{1}{\sqrt{4}} \delta_i \left[ G_{\text{fol}}(j\omega) \right] \quad (IV.26)$$

So, to make the SV's at high frequencies, we want to have
\[ G_p \hat{z}_y = I \quad (11.27) \]

or

\[ v = G_p (G_p \xi_p)^{-1} \quad (11.28) \]

**Numerical Example**

Let's look at a 4-input, 4-output control system for a submarine. The subs dynamics were linearized at 30 knots, and it is desired to independently and simultaneously control the submarine to follow commands. Specifically:

1. The depth rate
2. The pitch attitude
3. The heading rate, and
4. The roll angle

This is accomplished by dynamically coordinating the following control surfaces:

1. The bow fairweather plane
2. The rudder
3. Starboard stern plane
4. Port stern plane.
Figure 1 shows the SV's of the scaled plant model. Figure 2 shows the SV's of the plant augmented with four integrators at the control channels. Figure 3 shows the SV's of $\frac{1}{\sqrt{F}} G_{pf}(s)$ matched at low and high frequencies using exactly the formulas suggested. Figure 4 shows the SV's of $G_{pf}(s)$. Figure 5 shows the recovered loop singular values that one obtains through the use of the LQG/LTR compensator, while Fig. 6 shows the closed-loop singular values.
Fig. 1. SV's of open loop plant (scaled).

Fig. 2. SV's of plant augmented with integrators.
Fig. 3. Kalman Filter Open loop $G_{of}$ also ~ Kalman Filter loop $G_{KF}$

Fig. 5 Recovered Open Loop Transfer Function $G(s)K(s)$, with $g = 1000$
Fig. 6. Closed-loop singular values.
Typical gain-schedule usually involves:

1. Selection of several operating operating points that cover the entire range of plant dynamics.

2. At each operating point, construct a linear time-invariant model of the plant and design linear compensators for each plant.

3. In between the operating points, the compensator gains are interpolated, or scheduled, resulting in a global compensator.

Because the local designs are based on linear models, the designer can guarantee that at each operating point the feedback system has robust stability, robust performance, and of course, nominal stability. However, for the gain-scheduled system, not even nominal stability can be guaranteed. Extensive simulations are usually performed to investigate the performance of the GSC system. There are a few rules of thumb, however:

1. The scheduling variable should vary slowly, and
2. The scheduling variable should capture the plant's non-linearities.
A. Scheduling on a Pre-Defined Trajectory

Assume that the dynamics of the nonlinear model be given by:

\[
\dot{x}(t) = f(x(t)) + B u(t), \quad x(0) = x_0, \quad \in \mathbb{R}^n
\]

\[
y(t) = C x(t)
\]

and assume that a reference trajectory \( \dot{x}_r(t) \) is generated by the control input \( u_r(t) \). For this input,

\[
\dot{x}_r(t) = f(x_r(t)) + B u_r(t), \quad x_r(0) = x_0, \quad \in \mathbb{R}^n
\]

\[
y_r(t) = C x_r(t)
\]

Now, define:

\[
\delta x(t) = x(t) - x_r(t)
\]

\[
\delta y(t) = y(t) - y_r(t)
\]

\[
\delta y_r(t) = y_r(t) - y_r(t)
\]

The linearized model dynamics (4) around the trajectory \( x_r(t) \) are:

\[
\delta \dot{x}(t) = \delta f(x^*(t), \delta x(t)) + B \delta y(t) + \delta f(t, x(t), x^*(t)) \tag{2}
\]

\[
\delta x(0) = x_0 - x_r^*
\]

\[
\delta y(t) = C \delta x(t) \tag{3}
\]
where,
\[
\delta f(t, x(t), x^*(t)) = f(x(t)) - \mathcal{F} \left[ f(x^*(t)) + \delta f(x^*(t)) \delta x(t) \right]
\]

Now, define the time-frozen plan model:
\[
\delta \dot{x}(t) = \mathcal{D} f \left( x^*(t) \right) \delta x(t) + B \delta y(t) \tag{4}
\]
\[
\delta x(0) = x_0 - x_0^*, \quad t \in \mathbb{R}^+
\]
\[
\delta y(t) = C \delta x(t) \tag{5}
\]

We can consider designing for a known trajectory $r^*$. In this case, the time-invariant design plant (4), (5) can be considered as a frozen-parameter plant with parameter $t$. It would be however, impractical to require design for each $t$. So, practical considerations would limit this approach to either finite horizon control problems or "asymptotically" constant/periodic reference trajectories.

If the reference trajectory $r^*(t)$ is not known, then the design must be performed for families of potential reference trajectories.

Again, nothing can be guaranteed for the G5. design.
Consider the following plant:

\[
\frac{d}{dt} \begin{pmatrix} y \\ z \end{pmatrix}(t) = \begin{pmatrix} f(y(t), z(t)) + Bu(t) \\ v(t) \end{pmatrix}
\]

where the plant output \( y \) is a state variable.

Assume that there is a family of equilibrium conditions that can be parameterised in terms of the output \( y \). (For gain-scheduling each equilibrium condition is a possible operating point). So, assume that:

\[
0 = f(y, \mu_{eq}(y)) + Bu_{eq}(y)
\]

or,

\[
u_{eq} = H(y)
\]
Let's linearize the plant (6) around an possible operating point \( y_0 \), as follows:

\[
\frac{d}{dt} \begin{pmatrix} y(t) - y_0 \\ z(t) - z_{ep}(y_0) \end{pmatrix} = \mathcal{D} \begin{pmatrix} y(t) - y_0 \\ z(t) - z_{ep}(y_0) \end{pmatrix} + B \begin{pmatrix} u(t) - u_{ep}(y_0) \end{pmatrix} \tag{7}
\]

For each operating point, we design a compensator based on the local LTI model (7). This results in a family of compensators given by:

\[
\left( A_k(y_0), B_k(y_0), C_k(y_0) \right)
\]

parametrized by the operating condition \( y_0 \).

As seen in the figure, the operating condition is instantaneously updated using the plant output, and this in turn is used to update the compensators, which evolve as:

\[
\dot{x}_k(t) = A_k(y(t)) x_k(t) + B_k(y(t)) u(t)
\]

\[
\mathcal{S}_k(t) = C_k(y(t)) x_k(t)
\]

For each frozen compensator, i.e. \( \mathcal{S}_k(t) = y_0 \), the equivalent system has desirable properties.
Proofs:

**Theorem 1:** Assumptions: (1) Square system
(2) Stabilizable & Detectable
(3) Strictly min. Phase.

Then, \( \lim_{p \to 0} K_p = 0 \)

\( K_p \) solution of CARE. (See Kwackernau & Simon pp. 306-312).

**Theorem 2:** Assumptions (1), (2), (3) & consider the cheap LD-problem, as \( p \to 0 \). Then,

\[
\lim_{p \to 0} \int_0^p G_p \to W C \quad ; \quad W W = I
\]

\[
G_p = \frac{1}{p} B' K_p
\]

Assume,

\[
0 = -K A - K B' K_p - C' C + \frac{1}{p} K_p B B' K_p
\]

and consider the limiting case as \( p \to 0 \), using Th. 1,

\[
-C' C + \frac{1}{p} K_p B B' K_p \to 0
\]

Also,

\[
\hat{G}_p = \frac{1}{p} B' K_p \Rightarrow
\]

\[
\sqrt{p} \ G_p = \frac{1}{\sqrt{p}} B' K_p \quad \text{so}
\]
$$\left( \sqrt{p} \ G_p \right) \left( \sqrt{p} \ G_p \right) \rightarrow C \ C \ \text{as} \ \ p \rightarrow 0$$

which implies that

$$\sqrt{p} \ G_p \rightarrow WC \ \text{where}, \ \ W \ W = I.$$ 

**Lemma 1:**

$$k_p \ (s) = G_p \left( s \ I - A + BG_p + WC \right)^{-1}$$

such that

$$\Re \ \eta_i \ [A - WC] < 0$$

$$\Re \ \eta_i \ [A - BG_p] < 0 \ \text{for} \ 0 < p < \infty$$

such that

$$\lim_{p \rightarrow 0} \sqrt{p} \ G_p = WC \ ; \ W \ W = I$$

Then,

$$\lim_{p \rightarrow 0} k_p \ (s) = \left[ C \left( s \ I - A \right)^{-1} B \right]^{-1} \left( s \ I - A \right)^{-1}$$

pointwise convergence in $s$.

**Proof:**

It is true that:

$$\left( A + BCD \right)^{-1} = A^{-1} - A^{-1} B \left( C + DA^{-1} B \right)^{-1} DA^{-1}$$

Now, define, \( \Phi(s) = (sI - A)^{-1} \), \( \Phi^{-1} = (sI - A) \)
\[ X(s) = \left( \Phi^{-1}(s) + HC \right)^{-1}; \quad X^{-1}(s) = \Phi^{-1} + HC \]

The, MBC, TF is,

\[ X(s) = G_p \left( sI - A + BG_p + H \right)^{-1} H \]

\[ = G_p \left( \Phi^{-1}(s) + HC + BG_p \right)^{-1} H \]

\[ = G_p \left( X^{-1}(s) + BG_p \right)^{-1} H \]

Using the above identity:

\[ K_p(s) = G_p \left[ X(s) - X(s) B (I + G_p X(s) B)^{-1} G_p X(s) \right] H \]

\[ = \left[ G_p X(s) - G_p X(s) B (I + G_p X(s) B)^{-1} G_p X(s) \right] H \]

\[ = \left[ I - G_p X(s) B (I + G_p X(s) B)^{-1} \right] G_p X(s) H \]

I = \left( I + G_p X \right) \left( I + G_p X \right)^{-1}

Substitute b factor

\[ K_p(s) = \left[ I + G_p X B - G_p X(s) B \right] \left[ I + G_p X \right]^{-1} G_p X H \]

which yields,

\[ K_p(s) = \left( I + G_p X(s) B \right)^{-1} G_p X H \]

Multiplying & dividing by \( G_p \) we obtain:
\[ K_p(s) = \left( I + \sqrt{p} G_p \times (s) B \right)^{-1} \sqrt{p} G_p \times (s) H \]

So, \( p \to 0 \),

\[ \lim_{p \to 0} K_p(s) = \left( W \subseteq X(s) \times (s) B \right)^{-1} W \subseteq X(s) H = \]

\[ = \left( \subseteq X(s) \times (s) B \right)^{-1} W^{-1} W \subseteq X(s) H \]

\[ = \left( \subseteq X(s) \times (s) B \right)^{-1} \subseteq \Phi(s) H \]

It can be shown that:

\[ \subseteq X(s) H = \left( I + \subseteq \Phi(s) H \right)^{-1} \subseteq \Phi(s) H \] (using the matrix identity)

&

\[ \subseteq X(s) B = \left( I + \subseteq \Phi(s) H \right)^{-1} \subseteq \Phi(s) B \]

So,

\[ \lim_{p \to 0} K_p(s) = \left( \subseteq \Phi(s) B \right)^{-1} \left( I + \subseteq \Phi(s) H \right) \left( I + \subseteq \Phi(s) H \right)^{-1} \subseteq \Phi(s) H \]

\[ = \left[ \subseteq (S-I)^{-1} B \right]^{-1} \subseteq (S-I)^{-1} H \]
What is left to cover:

\[
\lim_{s \to 0} \frac{G^{-1}(0) \frac{I}{s} G(s) + G_n^{-1}(s) G(s)}{s} = \frac{I}{s} \neq 0
\]

\[
G(0) = G(-A)^{-1}B
\]

\[
\lim_{s \to \infty} \frac{G^{-1}(0) \frac{I}{s} G(s) + G_n^{-1}(s) B(s)}{s} = 0 + \frac{CB}{s} \text{ if } G_n^{-1}(s) = \frac{I}{s}
\]

\[
G_{0e} = M (sI - A_k)^{-1} B_e
\]

\[
M = \begin{bmatrix}
(CB)^{-1} & (-CA^{-1}B)^{-1}
\end{bmatrix}
\]