1) Solution:

The equation of the half-wave is given by

\[ x(t) = \begin{cases} \cos(\omega t), & T/4 \leq t \leq T/4 \\ 0, & \text{otherwise} \end{cases} \]

The Fourier transform of a continuous signal is given by the formula

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

Therefore substituting the value of ‘x(t)’ in the above relation, we get

\[ X(\omega) = \int_{T/4}^{T/4} \cos(\omega t) e^{-j\omega t} dt = \int_{T/4}^{T/4} \left( \frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) e^{-j\omega t} dt = \int_{T/4}^{T/4} \left( \frac{1 + e^{-2j\omega t}}{2} \right) dt = \frac{1}{2} \left( \frac{T}{2} + \frac{e^{j\pi} - e^{-j\pi}}{2j\omega} \right) = \frac{T}{4} \]

2) Solution:

\[ F(\omega) = \int_{-\infty}^{\infty} U(x) Ae^{-j\omega x} e^{-j\omega x} dx = \int_{0}^{\infty} Ae^{-(a+j\omega)x} dx = A e^{-(a+j\omega)x} \bigg|_{0}^{\infty} = \frac{A}{(a+j\omega)} \]

3) Solution:

The inverse Discrete Fourier Transform is given by

\[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \]

Substituting \( X(\omega) \) from the definition yield

\[ x(n) = \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} e^{j\omega n} d\omega = \frac{1}{2\pi} \left( e^{j\omega n} \right)_{-\pi/3}^{\pi/3} = -\frac{2}{\pi} \left( e^{j\omega n} - e^{-j\omega n} \right) = \frac{2\sin(n\pi/3)}{n\pi/3} \]

4) Solution:

The Laplace transform of the function \( f(t) = \sin(t) \) is given by \( F(s) = \frac{1}{s^2 + 1} \).

Therefore using the identity \( L[tf(t)] = -\frac{d}{ds} F(s) \), we have

\[ L[t \sin(t)] = -\frac{d}{ds} \left( \frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2} \]
So the Laplace transform of the function \( f(t) = 1 + t \sin(t) \), is

\[
F(s) = \frac{1}{s} + \frac{2s}{(s^2 + 1)^2}.
\]

5) Solution:

The Laplace transform of a function \( f(t) \) is given by

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt.
\]

Therefore substituting the value of the function in the above relation we get

\[
F(s) = \int_0^\infty te^{-at}e^{-st} \, dt = \int_0^\infty e^{(a+s)t} \, dt.
\]

Integrating by parts, we get

\[
F(s) = \left[ \frac{te^{-(a+s)t}}{-(a+s)} \right]_0^\infty + \frac{1}{(a+s)} \left[ e^{-(a+s)t} \right]_0^\infty = \frac{1}{(s+a)^2}.
\]

6) Solution:

a. The frequencies present in the signal are

\[
2\pi F_1 = 500\pi \Rightarrow F_1 = 250 \text{ Hz},
\]

\[
2\pi F_2 = 1000\pi \Rightarrow F_2 = 500 \text{ Hz},
\]

\[
2\pi F_3 = 2000\pi \Rightarrow F_3 = 1000\text{Hz}.
\]

Therefore \( F_{max}=1000 \) Hz and according to the sampling theorem the minimum sampling frequency is given by

\[
F_s > 2F_{max} \Rightarrow F_s > 2000 \text{ Hz}.
\]

b. If the sampling frequency is 1000Hz, then according to the sampling theorem the maximum frequency that can be recovered from the discrete signal is given by

\[
F_s/2=500 \text{ Hz}
\]

7) Solution:

a. (1) Signal sampled at 200Hz for 1 second.
It can be seen that there are three frequency components at 10 Hz, 70 Hz and 80 Hz respectively.

Note that the magnitude of the 70 Hz component is 3 times that of the 10 Hz and 80 Hz components.

(2) Signal sampled at 200Hz for 10 seconds.
(3) Signal sampled at 200Hz for 100 seconds.
It can be seen that as the number of data points increase, the side lobes in the magnitude plot decrease considerably.

b. For case (3) of part(a), signal downsampled by a factor of 2. The plot is shown in below
Since the sampling frequency is 100 Hz, only frequencies up to 50 Hz can be seen without any aliasing. The remaining frequencies appear as aliased frequencies. Therefore the 10 Hz component appears without any aliasing, whereas the 70 Hz and 80 Hz components appear as 20 Hz and 30 Hz components, i.e.,

\[ F_{\text{aliased}} = |F_{\text{sampling}} - F_{\text{signal}}|, \]

Therefore,

\[ F_1 = |100 - 70| = 30 \text{ Hz}, \]
\[ F_2 = |100 - 80| = 20 \text{ Hz}. \]

c. Therefore a low pass filter is needed before downsampling.

The resampled result is shown in below
Mablab Codes:

For part (a)

1. \( \text{time} = 1; \)
   \[
   t = 0:1/200:\text{time};
   \]
   \[
   x = \sin(20\pi t) + 3\sin(140\pi t) + \cos(160\pi t);
   \]
   \[
   f = \text{fft}(x, 8192);
   \]
   \[
   \text{freq} = -100:200/8192:100-200/8192;
   \]
   \[
   \text{plot(freq, abs(fftshift(f)))};
   \]
   \[
   \text{grid}
   \]
   \[
   \text{xlabel('Frequency (Hz)')};
   \]
   \[
   \text{ylabel('Magnitude')};
   \]
   \[
   \text{title('Original signal sampled at 200Hz for 1 second')};
   \]

2. \( \text{time} = 10; \)
t = 0:1/200:time;
  x = sin(20*pi*t)+3*sin(140*pi*t)+ cos(160*pi*t);
  f = fft(x, 8192);
  freq = -100:200/8192:100-200/8192;
  plot(freq, abs(fftshift(f)));
  grid
  xlabel('Frequency (Hz)');
  ylabel('Magnitude');
  title('Original signal sampled at 200Hz for 10 second');

(3) time = 100;
  t = 0:1/200:time;
  x = sin(20*pi*t)+3*sin(140*pi*t)+ cos(160*pi*t);
  f = fft(x, 8192);
  freq = -100:200/8192:100-200/8192;
  plot(freq, abs(fftshift(f)));
  grid
  xlabel('Frequency (Hz)');
  ylabel('Magnitude');
  title('Original signal sampled at 200Hz for 10 second');

For Part (b) and Part(c):
  time = 100;
  t = 0:1/200:time;
  x = sin(20*pi*t)+3*sin(140*pi*t)+ cos(160*pi*t);
  x1 = dyaddown(x);
  x2 = resample(x,1,2);
  f1 = fft(x1, 8192);
  f2 = fft(x2, 8192);
  freq1 = -50:100/8192:50-100/8192;
  figure(1)
  plot(freq1, abs(fftshift(f1)))
grid
xlabel('Frequency (Hz)');
ylabel('Magnitude');
title('Signal downsampled by a factor of 2');
figure(2)
plot(freq1, abs(fftshift(f2)))
grid
xlabel('Frequency (Hz)');
ylabel('Magnitude');
title('Signal resampled by a factor of 2');