I. We now introduce a class of MIMO compensators which are called Model-Based Compensators (MBC) which have the property that when put together with any MIMO open-loop plant, the resultant closed-loop system will be stable, provided that we select the gain parameter appropriately. We will analyze the behavior of the closed-loop system when using MBC's. The mathematical analysis shows that MBC's can be constructed using results from optimal control theory (so-called linear-quadratic feedback problem) and from optimal estimation theory (the so-called Kalman filter). As a matter of fact, the so-called linear-quadratic Gaussian (LQG) compensators are a subclass of MBC's.

The problem that a control system designer faces are important considerations in designing MIMO compensators. Given a possible unstable open-loop plant, the designer must select and implement a compensator that:

1. Attains and/or maintains stability of the closed-loop system.

2. Achieves desirable loop-shapes in the frequency domain using appropriate singular values.
Up to this point we don't have a systematic procedure for achieving the stability objective given an arbitrary MIMO open-loop plant. We have the necessary tools for analyzing stability both in the time-domain (by matrix eigenvalues) and the frequency domain (by the multivariable Nyquist stability theory). However, we have not been able to extend, in an easy and natural manner, the SISO design concepts of root locus, Nyquist diagram etc., to the MIMO case in a systematic manner. That is, we still do not know how to design a MIMO compensator \( K(s) \) for a feedback system, as follows:

![Feedback structure of a MIMO control system with disturbances \( d(s) \) reflected at the plant input.]

Fig 1. Feedback structure of a MIMO control system with disturbances \( d(s) \) reflected at the plant input.

given the nominal plant transfer matrix \( G_p(s) \), so that the resultant closed-loop system is stable, and we still have enough degrees of freedom for loop-shaping to achieve performance and maintain robustness.

Now we define a very special class of MIMO compensators, i.e., MBC's, in order to meet our mimo objectives. These compensators contain the dynamics of the
Fig. 2. The Model Based Controller in a Feedback Configuration.
nominal open-loop plant \( G_p(s) \) in an explicit way. In addition, MBC's do contain several degrees of freedom incorporated in two gain matrices, which can be adjusted to achieve the necessary performance-robustness tradeoffs while maintaining closed-loop stability.

\[ \text{Remark:} \quad \text{The MBC structure has been derived from continuous progress in the area of pure optimal estimation and control theory. However, the word "optimal" should not be taken out of context because the results we will obtain, i.e., the compensator \( K(s) \), will not be optimal in any sense.} \]

\[ \text{II. The MBC:} \]

Figure 2 shows an MBC internal structure along with an open-loop plant \( G_p(s) \). The word MBC is because the matrices A, B, C of the open-loop plant appear also in the dynamics of the open compensator in a very specific way, almost duplicating the input-state-output interconnections in the plant.

\[ \text{The open-loop dynamics} \]

We will assume that the open-loop dynamics can be described by the following vector differential equation:

\[ \]
\[ \dot{x}(t) = A x(t) + B u(t) + L d(t) \]  
(II.1)
\[ y(t) = C x(t) \]  
(II.2)

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( d(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^m \).

The control to output transfer matrix is:
\[ G_p(s) = C (sI - A)^{-1} B \quad \text{so that,} \quad y(s) = G_p(s) u(s) \]  
(II.3)

And the disturbance to output relation is:
\[ y(s) = C (sI - A)^{-1} B + d(s) \]  
(II.4)

So, that
\[ y(s) = C (sI - A)^{-1} B y(s) + C (sI - A)^{-1} d(s) \]  
(II.5)

The MBC Dynamics

The vector \( \bar{z}(t) \) is the state vector of the MBC and \( \bar{z}(t) \in \mathbb{R}^n \), i.e. \( \bar{z}(t) \) has the same dimension as the plant state \( x(t) \). Thus the MBC is an nth order system. The dynamics of the MBC can be written as:
\[ \dot{\bar{z}}(t) = A \bar{z}(t) + B u(t) + L v(t) \]  
(II.6)
\[ v(t) = -\bar{z}(t) - C \bar{z}(t) = y(t) - C \bar{z}(t) - \bar{z}(t) \]  
(II.7)
\[ u(t) = -G \bar{z}(t) \]  
(II.8)
Substituting (7) and (8) into (6) we have:

\[ \ddot{x}(t) = [A - BG - HC] \dot{x}(t) - HE(t) \]  \((II-9)\)

The input to the MBC is the error \( e(t) \) and the output of the MBC is the control \( u(t) \). Thus (3) \& (7) constitute an input-state-output description of the MBC. Therefore the MBC transfer function \( K(s) \) defined by

\[ H(s) = K(s) E(s) \]  \((II-10)\)

is given by:

\[ K(s) = G (sI - A + BG + HC)^{-1} H \]  \((II-11)\)

Note that the two gain matrices \( G \) and \( H \) which represent the design parameters of the MBC, in effect change the poles and zeros of the \( K(s) \).

### III. Closed-Loop System Dynamics

We can now calculate the closed-loop dynamics of the system either in the time or frequency domain. Apparently the time-domain analysis reveals with more insight.
A Closed-loop Representation

By substituting eqs. 3.8 into 3.7 we obtain:

\[
\begin{align*}
\dot{x}(t) &= A x(t) - B \xi(t) + \zeta(t) \\
y(t) &= C x(t)
\end{align*}
\] (III-1)  

Since,

\[
e(t) = r(t) - y(t) = r(t) - C x(t)
\] (III-3)

Then, eq. (II-9) yields:

\[
\ddot{z}(t) = H C x(t) + [A - BG - HC] z(t) - H r(t)
\] (III-4)

Note eq. (III-1) and (III-4) represent a system of \(2m\) LTI differential equations; they can be written as:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{z}(t)
\end{bmatrix} =
\begin{bmatrix}
A & -BG \\
HC & A - BG - HC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
z(t)
\end{bmatrix} +
\begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
\xi(t) \\
r(t)
\end{bmatrix}
\] (III-5)

Eq. (III-5) describes the complete closed-loop system. Given \(A, B, C\) a particular selection of the gain matrices \(C\) and \(H\) will completely specify the closed-loop dynamics from which we can calculate the closed-loop response to disturbance and/or command inputs.
The stability of the closed-loop system will be ensured if

\[ \text{Re} \lambda_i [A_{cl}] < 0 \quad i = 1, \ldots, 2m. \quad (\text{III-6}) \]

However, the structure of the $2m \times 2m$ matrix $A_{cl}$ is not very transparent in (III-5), in order to deduce whether there exist matrices $G$ and $H$ that will imply equation (III-6).

A more transparent Closed-loop Representation

The features of the closed-loop system of figure 2 can be made more transparent if we make a change of variables, which corresponds to a different state space of the closed-loop dynamics.

The appropriate change of variables is:

\[ W(t) = X(t) - \frac{\dot{z}(t)}{f(t)} \quad (\text{III-2}) \]

It follows that:

\[ \dot{W}(t) = \dot{X}(t) - \frac{\ddot{z}(t)}{f(t)} = \]

\[ = (A - HC) W(t) + L \phi(t) + U(t) \quad (\text{III-8}) \]

Therefore, the dynamics of the vector $W(t)$ are not coupled to the vector $X(t)$. Also (III-1) and (III-2) result in:

\[ \dot{X}(t) = (A - BG) X(t) + BG W(t) + L \phi(t) \quad (\text{III-9}) \]
Thus, the dynamics of the closed-loop system are given by:
\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{w}(t)
\end{bmatrix} =
\begin{bmatrix}
A - BG & BG \\
0 & A - HC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix} +
\begin{bmatrix}
L & 0 \\
L & H
\end{bmatrix}
\begin{bmatrix}
d(t) \\
f(t)
\end{bmatrix}
\] (III-10).

Using the determinant identity:
\[
\det \begin{bmatrix}
x & y \\
0 & z
\end{bmatrix} = (\det x)(\det z),
\]
the \(n\) eigenvalues of the closed loop system are the roots of the equation:
\[
0 = \det \begin{bmatrix}
A - A + BG & -BG \\
0 & A - A + HC
\end{bmatrix}
\]
\[
= \det (A - A + BG) \cdot \det (A - A + HC) \tag{III-11}
\]
Now, let
\[
\phi_1 (\lambda, G) = \det (A - A + BG) \tag{III-12}
\]
\[
\phi_2 (\lambda, H) = \det (A - A + HC) \tag{III-13}
\]
In order for the closed-loop system to be stable, we must have:
\[
\text{Re } \lambda_i [A - BG] < 0, \quad i = 1, \ldots, n \tag{III-14}
\]
And
\[
\text{Re } \lambda_i [A - HC] < 0, \quad i = n + 1, n + 2, \ldots, 2n \tag{IV-15}
\]
Then the stability of the closed-loop system decomposes into two separate problems.

**Problem 1.** Given \( A \) and \( B \) can we find a \( \xi \) such that

\[
\text{Re} \left[ \text{Di} \left[ A - BC \right] \right] < 0 \quad i = 1, \ldots, n?
\]

**Problem 2.** Given \( A \) and \( C \) can we find an \( H \) such that

\[
\text{Re} \left[ \text{Di} \left[ A - HC \right] \right] < 0 \quad i = 1, 2, \ldots, m?
\]

It turns out that results available from linear system theory can be used to answer these problems in the affirmative. Let's now summarize these results.

**IV. Results from Linear System Theory**

**Definition IV.1:** The system

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \quad (IV-1)
\]

is controllable, or \([A, B]\) is a controllable pair, iff

\[
\text{rank} \left[ B, AB, \ldots, A^{n-1}B \right] = n \quad (IV-2)
\]

The system \((IV-1)\) is stabilizable, or \([A, B]\) is a stabilizable pair, if the uncontrollable modes of the system are stable. Note that controllability implies stabilizability.
Theorem III. 4 Given the system

\[ \dot{x}(t) = A x(t) + E u(t) \quad (\text{IV}-3) \]

Consider the full state variable feedback control law

\[ u(t) = -S x(t) \quad (\text{IV}-4) \]

resulting in the closed-loop system

\[ \dot{x}(t) = [A - BS] x(t) = Ac x(t) \quad (\text{IV}-5) \]

(a) If \([A, B]\) is a controllable pair then there exists at least one state feedback gain matrix \(S\) (there may be many) such that all of the closed-loop poles can be placed at arbitrary locations (subject to the complex conjugate) constraints. In particular, there exists at least one state feedback gain matrix \(S\) such that all closed-loop poles are at pre-specified locations in the left-half \(s\)-plane.

(b) If \([A, B]\) is only a stabilizable pair, then there exists at least one state feedback gain matrix \(S\) such that all closed-loop poles can be placed in the left-half \(s\)-plane.

Note: There are pole placement algorithms that give \([A, B]\) and desirable closed-loop poles, find the gain matrix \(S\). However, we'll not discuss these.

Comment: Theorem III. 1 gives an affirmative answer to Problem 4 posed above.
In addressing Problem 2, we state the following definition.

**Definition IV. 2:** The system,
\[ \dot{x}(t) = A \cdot x(t) \quad y(t) = C \cdot x(t) \quad x \in \mathbb{R}^n, \ y(t) \in \mathbb{R}^p \quad \text{(IV-6)} \]
is observable, or the pair \([A, C]\) is an observable pair, if,
\[ \text{rank} \left[ C, C A, \ldots, (C A)^{n-1} \right] = n \quad \text{(IV-7)} \]
The system \((IV-6)\) is detectable or \([A, C]\) is a detectable pair, if the unobservable modes of the system \((IV-6)\) are stable. Note that observability implies detectability.

**Theorem IV. 2:** Consider the system \((B=I)\)

\[ \dot{x}(t) = A \cdot x(t) + u(t) \quad x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^n \quad \text{(IV-8)} \]
\[ y(t) = C \cdot x(t) \quad y(t) \in \mathbb{R}^p \quad \text{(IV-9)} \]

(a) Suppose that \([A, C]\) is observable. Then there exists an output feedback gain matrix \(H\) (or maybe many)
\[ y(t) = -H \cdot y(t) = -H \cdot u(t) \quad \text{(IV-10)} \]
leading to the closed-loop system
\[ \dot{x}(t) = [A - HC] \cdot x(t) \quad \text{(IV-11)} \]
such that all the closed-loop eigenvalues $\lambda_i \{A - BC\}$ can be placed in prespecified locations.

(b) Suppose that $[A, C]$ is detectable. Then there exists at least one output gain matrix $H$ such that all closed-loop poles $\lambda_i \{A - HC\}$ of $(W, H)$ are in the left-half $s$-plane.

Note: This theorem can be proved by using the duality of controllability and observability.

Comment: Theorem (IV.2) basically answers Problem 2 in an affirmative fashion. This theorem is also central in the derivation of a stable

![Block diagram](image)

**Fig. IV.1** Visualization of the stable control system implied by Theorem IV.1 if $[A, B]$ is stabilizable under full state variable feedback.
Fig. IV.2 Visualization of the stable control system implied by Theorem IV.2 if $[A, C]$ is detectable under output feedback.

Fig. IV.3 Definition of the observer problem. If $[A, C]$ is detectable and $\text{Re} \chi_i [A - HC] < 0$ then,

$$ \lim_{t \to \infty} \hat{x}(t) = x(t), \quad \text{and} \quad \lim_{t \to \infty} y'(t) = y(t). $$

Note: The observer problem consists of deducing a dynamic system (observer) driven by the output $y(t)$ of the plant with the property that the observer state vector $\hat{x}(t)$ approaches the plant state $x(t)$ as $t \to \infty$. As shown, the observer involves a feedback loop of the form shown in Fig IV.2. The observer gain matrix $K$ is selected such that the observer dynamic are.
stable, i.e., \( \text{Re} \, \lambda_i \, [A-BC] < 0 \).

II. A “Cook Book” Calculation of the \( G \) and \( H \) Gain Matrices (LQG compensator)

We have shown that the feedback loop of Fig. 2 can be made closed-loop stable by the appropriate selection of the two gain matrices \( G \) and \( H \).

Now we present a procedure for actually evaluating numerical values for \( G \) and \( H \) using available computer tools. This procedure is based on the LQG stochastic optimal control theory. Therefore, LQG-based compensators are a special class of MBC’s.

VI. Calculating the \( G \) gain matrix (control gain) using the Linear-Quadratic (LQ) algorithm.

Let’s start with the \( m \times m \) matrix \( A \) of \( n \times n \) matrix \( B \); we assume that \([A,B] \) is stabilizable.

Step 1: Select an arbitrary \( m \times m \) symmetric and positive definite matrix denoted by \( R \), i.e., \( R = R' > 0 \) (V.1)

Step 2: Select an arbitrary \( p \times n \) (\( p \leq n \)) matrix denoted by \( N \) with the only property that \([A,N] \) is detectable.
Let 
\[ \mathbf{Q} = \mathbf{N} \mathbf{N}^{T} \] 
which has the property 
\[ \mathbf{Q} = \mathbf{Q}^{T} > 0 \] 
\( (V.2) \) 
\( (V.3) \)

Remark: \( \mathbf{N} \) is called a square root matrix of \( \mathbf{Q} \).

**Step 3:** Let the computer subroutines solve for the unique positive semi-definite symmetric \( m \times m \) solution matrix \( \mathbf{K} \), of the so-called control algebraic Riccati equation (CARE).

\[-\mathbf{K} \mathbf{A} - \mathbf{A}^{T} \mathbf{K} - \mathbf{Q} + \mathbf{K} \mathbf{BR}^{\dagger} \mathbf{B}^{T} \mathbf{K} = 0 \] 
(People usually pick \( \mathbf{Q} = \mathbf{C} \mathbf{C}^{*} \) and \( \mathbf{R} = \mathbf{I} \))

**Step 4:** Let also the computer subroutines calculate the \( m \times n \) gain matrix \( \mathbf{G} \) by the formula
\[ \mathbf{G} = \mathbf{R}^{\dagger} \mathbf{B} \mathbf{K} \] 
\( (V.5) \)

Remark: The assumptions of the above steps guarantee that:
\[ \text{Re} \, \text{Di}[\mathbf{d} - \mathbf{BG}] < 0; \quad i = 1, 2, \ldots, n \] 
\( (V.6) \)
No matter what are the numerical values of \( m, n, d, B \) \( \mathbf{G} \) and \( \mathbf{K} \).

**V.2 Calculating the \( H_{2} \) gain matrix (filter gain) using the Kalman Filter algorithm (KF).**

We start with the \( m \times n \) matrix \( \mathbf{G} \) and the \( m \times n \)
matrix $\Xi$; we assume that $[A, E]$ is detectable.

**Step 1:** Select an arbitrary $m \times m$ symmetric and positive definite matrix denoted by $\Theta$; i.e.

$$\Theta = \Theta' > 0 \quad (v.7)$$

**Step 2:** Select an arbitrary $m \times p$ matrix $M$ with the only property that $[A, M]$ is stabilizable. Let $\Xi$ denote the $m \times m$ matrix

$$\Xi = MM' \quad (v.8)$$

which has the property

$$\Xi = \Xi' > 0 \quad (v.9)$$

Note that $\Xi'$ is the square root of $\Xi$.

**Step 3:** Let the computer subroutines solve for the unique positive-semidefinite symmetric $m \times n$ solution matrix, $\Xi$ of the so-called filter algebraic matrix Riccati equation (FARE).

People usually pick $\Xi = LL'$ and $\Theta = I$

$$A'\Xi + \Xi A' + \Xi - \Xi C' \Theta^{-1} C \Xi = 0 \quad (v.10)$$

**Step 4:** The computer subroutines will also calculate the $m \times n$ gain matrix $H$ by:

$$H = \Xi C' \Theta^{-1} \quad (v.11)$$

Remarks: The assumptions and the above steps guarantee that:

$$\text{Re} \, \Rei \left[ A - H C \right] < 0, \quad i = 1, 2, \ldots, n \quad (v.12)$$
no matter what are the numerical values of \( n, m, \alpha, \beta, \epsilon, \theta \)

Summary: When the gain matrices \( G \) and \( H \) of the HNC are calculated by the above procedure, then we refer to the HNC as a Linear Quadratic Gaussian (LQG) compensator. We show that an LQG compensator guarantees the closed-loop stability of the feedback control system because the properties (V6) and (V.12) are guaranteed and \( \tau_i \left[ A - BG \right] \) \& \( \tau_i \left[ A - HC \right] \) define the \( 2n \) closed-loop poles.

**VII. The Closed-Loop Transfer Matrices**

**Input to output:**

\[
y(s) = G_{cl}(s) r(s)
\]  \hspace{1cm} (VII.1)

\[
y(s) = \frac{C (sI - A + BG)}{(sI - A + BG)(sI - A + HC)} H r(s)
\]  \hspace{1cm} (VII.2)

\[
G_{cl}(s)
\]

**Disturbance to output:**

\[
y(s) = G_{cl}^0(s) d(s)
\]  \hspace{1cm} (VII.3)

\[
y(s) = \frac{C (sI - A + BG)^{-1} \left[ I - BG(sI - A + HC)^{-1} \right]}{G_{cl}^0(s)} d(s)
\]  \hspace{1cm} (VII.4)